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ON THE NUMBER OF RESTRICTED PRIME FACTORS OF AN INTEGER. III

by Karl K. NORTON

§1. INTRODUCTION

Let P be the set of all (positive rational) prime numbers, and let E be an arbitrary nonempty subset of P . Throughout this paper, let p denote a general member of P , and for non-negative integers a , write $p^a \parallel n$ if $p^a \mid n$ and $p^{a+1} \nmid n$. For each positive integer n , define

$$\omega(n; E) = \sum_{p \mid n, p \in E} 1, \quad \Omega(n; E) = \sum_{p^a \parallel n, p \in E} a.$$

We usually write $\omega(n; P) = \omega(n)$, $\Omega(n; P) = \Omega(n)$. In this paper, we shall estimate the functions

$$S(x, y; E, \omega) = \text{card} \{n \leq x : \omega(n; E) > y\}, \tag{1.1}$$

$$S(x, y; E, \Omega) = \text{card} \{n \leq x : \Omega(n; E) > y\}$$

when y is appreciably larger than the normal order of $\omega(n; E)$ and $\Omega(n; E)$; y may even be as large as the maximum order of $\omega(n; E)$ or $\Omega(n; E)$, respectively. (Here and throughout, $\text{card } B$ means the number of members of the set B , and if $Q(n)$ is a statement about the integer n , we often write $\{n \leq x : Q(n)\}$ instead of $\{n : 1 \leq n \leq x \text{ and } Q(n)\}$.)

Define

$$E(x) = \sum_{p \leq x, p \in E} p^{-1} \quad (x \text{ real}). \tag{1.2}$$

In [13], it was observed that if $E(x) \rightarrow +\infty$ as $x \rightarrow +\infty$, then both the average order and the normal order of $\omega(n; E)$ are equal to $E(x)$, and the same statement holds for $\Omega(n; E)$. In [13], we obtained sharp inequalities for the functions (1.1) when $0 < y < 2E(x)$, roughly. In [14], we gave asymptotic formulas for the same functions when $E(x) \rightarrow +\infty$ and $y = E(x) + o(E(x))$ as $x \rightarrow +\infty$. It is well-known, however, that

$$E(x) \leq \log \log x + O(1) \quad \text{for } x \geq 2,$$

whereas if x is large, $\omega(n; E)$ and $\Omega(n; E)$ may be much larger than $\log \log x$ for some values of $n \leq x$. For example, the method of [6, pp. 262-263, 359] shows that

$$\limsup_{n \rightarrow +\infty} \frac{\omega(n) \log \log n}{\log n} = 1, \quad (1.3)$$

and a more precise version of (1.3) was obtained in [12, pp. 96-100]. (See also the remarks at the beginning of §3 below.) Before stating estimates for the functions (1.1) when y is large, it seems worthwhile to generalize results like (1.3) to $\omega(n; E)$. First define

$$\pi(x; E) = \sum_{p \leq x, p \in E} 1 \quad (x \text{ real}), \quad (1.4)$$

and write

$$\begin{aligned} \log_2 x &= \log \log x, & \log_r x &= \log(\log_{r-1} x) \\ \text{for } r &= 3, 4, \dots \end{aligned} \quad (1.5)$$

THEOREM 1.6. *Suppose that there exists a real number $\gamma(E) > 0$ such that*

$$\begin{aligned} \pi(x; E) &= \gamma(E) (x/\log x) \{1 + O_E(1/\log x)\} \\ \text{for all } x &\geq 2. \end{aligned} \quad (1.7)$$

Then for each $n \geq 3$, we have

$$\begin{aligned} \omega(n; E) &\leq \frac{\log n}{\log_2 n} + \frac{\{1 + \log \gamma(E)\} \log n}{(\log_2 n)^2} \\ &\quad + O_E\left(\frac{\log n}{(\log_2 n)^3}\right), \end{aligned} \quad (1.8)$$

with equality for infinitely many n .

Here and throughout, the notation $O_{\delta, \varepsilon, \dots}$ implies a constant depending at most on $\delta, \varepsilon, \dots$, while O without subscripts implies an *absolute* constant. Likewise, for $i = 1, 2, \dots$, we shall write $c_i(\delta, \varepsilon, \dots)$ for a positive number depending at most on $\delta, \varepsilon, \dots$, while c_i will mean a positive absolute constant.

It is interesting to observe that a much weaker hypothesis than (1.7) still implies that the maximum order of $\omega(n; E)$ is approximately $(\log n) (\log_2 n)^{-1}$. See the remarks after the proof of Theorem 1.6 in §3.

After (1.3) and Theorem 1.6, it is natural to ask how often $\omega(n; E)$ and $\Omega(n; E)$ assume values appreciably larger than their normal order $E(n)$. It appears that rather little was known about this problem until very recently. The earliest contribution was by Hardy and Ramanujan [5] (reprinted in [15,

pp. 262-275]), whose estimate for $\text{card} \{n \leq x : \omega(n) = m\}$ leads easily to a good upper bound for $S(x, y; P, \omega)$ (essentially the same as the bound given in Theorem 1.14 below). However, they did not state explicitly a result of the latter type. For arbitrary E , much weaker upper bounds for $S(x, y; E, \omega)$ and $S(x, y; E, \Omega)$ can be derived from a general theorem of Turán [19] on the distribution of values of additive functions. (See also Turán [18] or Hardy and Wright [6, pp. 356-358] for the case $E = P$, and see [13, §§1, 3] and [14, pp. 18-19] for remarks on all of this early work.) For the particular functions $\omega(n; E)$ and $\Omega(n; E)$, Turán's bounds were improved considerably in the author's paper [13; (5.16), (5.15), (1.11)], where it was observed that for any set E ,

$$S(x, \alpha E(x); E, \omega) \leq x \exp \{(\alpha - 1 - \alpha \log \alpha) E(x)\} \quad (1.9)$$

for real $x \geq 1$, $\alpha \geq 1$, where $E(x)$ is defined by (1.2). A similar (slightly less precise) result was stated for $\Omega(n; E)$ when $1 \leq \alpha < p_1$, where p_1 is the smallest member of E . No lower bound was obtained in either case for $\alpha \geq 2$, so that the precision of (1.9) for large α was not clear. In a later paper [2], Erdős and Nicolas obtained a rather good estimate in the special case $E = P$. They showed that for any fixed α with $0 < \alpha < 1$,

$$\text{card} \{n \leq x : \omega(n) > \alpha (\log x) (\log_2 x)^{-1}\} = x^{1-\alpha+o(1)} \quad (1.10)$$

as $x \rightarrow +\infty$. (In fact, they obtained a somewhat more precise result resembling Theorem 4.13 below.) However, they did not get an analogous result for $\Omega(n)$, nor did they generalize to $\omega(n; E)$ or $\Omega(n; E)$. Furthermore, their method did not give good upper estimates for $S(x, y; P, \omega)$ when y is appreciably smaller than $(\log x) (\log_2 x)^{-1}$. We propose to remedy all of these drawbacks to some extent. First, we obtain the following lower bound by a refinement of the Erdős-Nicolas method:

THEOREM 1.11. *Suppose that there exists a real number $\gamma(E) > 0$ such that (1.7) holds. Let $\varepsilon > 0$, and suppose that $x \geq c_1(E, \varepsilon)$ and*

$$c_2(E) \leq y \leq (\log x) (\log_2 x)^{-1} + \{1 + \log \gamma(E) - \varepsilon\} (\log x) (\log_2 x)^{-2}. \quad (1.12)$$

Then

$$S(x, y; E, \omega) \geq x \exp \{-y (\log y + \log_2 y - \log \gamma(E) - 1) + O_E(y (\log_2 y) / \log y)\}. \quad (1.13)$$

(1.8) shows that only a very small weakening of the hypothesis (1.12) would be of any interest. In Theorem 3.20, we assume much less than (1.7) and derive a result similar to Theorem 1.11 (but somewhat weaker).

Concerning upper bounds for $S(x, y; E, \omega)$, we have obtained only a modest improvement of (1.9); see Theorem 4.8 and Corollary 4.12. It should be emphasized that (1.9) and Theorem 4.8 hold for an arbitrary set E (without the assumption (1.7)). Using the same methods, we deduce

THEOREM 1.14. *Suppose that there exists a real number $\gamma(E) > 0$ such that (1.7) holds. If $x \geq 3$ and $y \geq \gamma(E) \log_2 x$, then*

$$S(x, y; E, \omega) \leq x \exp \left\{ -y (\log y - \log_3 x - \log \gamma(E) - 1) - \gamma(E) \log_2 x + O_E(y/\log_2 x) \right\}. \quad (1.15)$$

Although there is a considerable gap between (1.13) and (1.15), the results are more general and somewhat sharper than those of Erdős and Nicolas [2]. In particular, we get a generalization of (1.10) (see Theorem 4.13). Theorems 1.11 and 1.14 also yield immediately the following result which could not be obtained by the Erdős-Nicolas method:

COROLLARY 1.16. *Suppose that there exists a real number $\gamma(E) > 0$ such that (1.7) holds. If $0 < \alpha < 1$ and $x \geq c_3(E, \alpha)$, then*

$$S(x, (\log x)^\alpha; E, \omega) = x \exp \left\{ -\alpha (\log x)^\alpha \log_2 x + O((\log x)^\alpha \log_3 x) \right\}.$$

It should be mentioned that when $E = P$ (the set of all primes) and $y/\log_2 x$ is bounded and not too close to 1, Theorems 1.11 and 1.14 can be replaced by a striking asymptotic formula which was recently obtained by H. Delange (for the proof, see [2]):

THEOREM 1.17 (Delange). *Let x, α, r_1, r_2 be real with $x \geq 3$, $1 < r_1 \leq \alpha \leq r_2$. Then*

$$S(x, \alpha \log_2 x; P, \omega) = \frac{F(\alpha) \alpha^{1/2 + \alpha \log_2 x - [\alpha \log_2 x]}}{(2\pi)^{1/2} (\alpha - 1)} \cdot \frac{x}{(\log x)^{1 - \alpha + \alpha \log \alpha} (\log_2 x)^{1/2}} \left\{ 1 + O_{r_1, r_2} \left(\frac{1}{\log_2 x} \right) \right\},$$

where $[z]$ means the largest integer $\leq z$ and

$$F(\alpha) = \frac{1}{\Gamma(\alpha+1)} \prod_p \left(1 + \frac{\alpha}{p-1}\right) \left(1 - \frac{1}{p}\right)^\alpha.$$

Delange obtained a similar result for card $\{n \leq x: \omega(n) \leq \alpha \log_2 x\}$ when $x \geq 3$, $(\log_2 x)^{-1} \leq \alpha \leq r_3 < 1$ (see [2]). In this connection, it is interesting to note the estimate

$$F(\alpha) = \exp \{-\alpha \log \alpha - \alpha \log_2 \alpha + (1-\gamma)\alpha + O(\alpha/\log \alpha)\}$$

for real $\alpha \geq 2$, where γ is Euler's constant. (Some effort is required to show this, and we omit the proof.)

For values of α near 1, Kubilius [8, Theorem 9.2] proved a result on the distribution of $\omega(n)$ which is similar to Theorem 1.17. His theorem was later extended by himself [9] and Laurinćikas [10] to somewhat more general additive functions, and it was generalized to $\omega(n; E)$ and $\Omega(n; E)$ by Norton [14]. The estimates for $S(x, y; E, \omega)$ derived in the present paper are not as precise as Theorem 1.17 or the earlier work cited, but they are more general with respect to E (except for [14]), and they hold for much larger values of y .

We now consider the function $\Omega(n; E)$. Here we assume that E is any nonempty set of primes; in particular, nothing like (1.7) is assumed. For completeness, we begin by stating the following easy result:

THEOREM 1.18. *Let p_1 be the smallest member of E . Then*

$$\Omega(n; E) \leq (\log n) (\log p_1)^{-1} \quad \text{for all } n \geq 1, \quad (1.19)$$

with equality if and only if $n = p_1^a$ for some integer $a \geq 0$.

This follows from

$$n \geq \prod_{p^a \parallel n, p \in E} p^a \geq \prod_{p^a \parallel n, p \in E} p_1^a = p_1^{\Omega(n; E)}.$$

We now proceed to estimate $S(x, y; E, \Omega)$ (defined by (1.1)). For $y \geq 2E(x)$, rather little previous work has been done on this problem, and all of it was restricted to the special case $E = P$ (the set of all primes). Selberg [17, p. 87] stated without detailed proof the following asymptotic formula:

$$\text{card } \{n \leq x: \Omega(n) = m\} \sim A 2^{-m} x \log x$$

for integers m satisfying $(2+\varepsilon) \log_2 x \leq m \leq B \log_2 x$. (Here $\varepsilon > 0$ is arbitrarily small, while A and B are positive absolute constants; it is not clear

from [17] how large B could be.) Selberg also gave an asymptotic formula for $\text{card} \{n \leq x : \omega(n) = m\}$ when $m/\log_2 x$ is bounded. His work was recently extended to considerably larger values of m (roughly $m < (\log x)^{3/5}$) by Kolesnik and Straus [7], whose theorems are quite complicated. These results, together with the formula

$$S(x, y; P, \Omega) = \sum_{y < m \leq Y} \text{card} \{n \leq x : \Omega(n) = m\} + S(x, Y; P, \Omega)$$

and different tools for estimating $S(x, Y; P, \Omega)$ from above, would yield some information about $S(x, y; P, \Omega)$. However, it appears that neither [17] nor [7] would thus lead to an estimate for $S(x, y; P, \Omega)$ which is both simple and reasonably precise when $y/\log_2 x$ is unbounded. To the best of our knowledge, the only previous result of the latter type is due to Erdős and Sárközy [3], who recently proved that

$$S(x, y; P, \Omega) \leq c_4 y^4 2^{-y} x \log x \quad \text{for } x \geq 3, y \geq 1. \quad (1.20)$$

We shall generalize their work to $S(x, y; E, \Omega)$ and get a sharper upper bound. Although the result could be phrased in terms of the function $E(x)$ (defined by (1.2)), it is more convenient to state it in terms of a real number v which in practice is taken to be an approximation to $E(x)$. (For example, if $E = P$, we could take $v = \log_2 x$.)

THEOREM 1.21. *Let x, v, y be real with $x \geq 1, v \geq 1$, and $y \geq 0$. Let p_1 be the smallest member of E , and define*

$$\Lambda = \Lambda(x, v; E) = \max \{2, |E(x) - v|\}. \quad (1.22)$$

Then

$$S(x, y; E, \Omega) \leq c_5 (p_1) p_1^{-y} x v^{1/2} e^{(p_1 - 1)v + p_1 \Lambda}. \quad (1.23)$$

We remark that (1.23) is our best upper bound when $y > p_1 v - v^{1/2}$, but it can be improved for smaller values of y (see Lemma 5.3).

Concerning the problem of estimating $S(x, y; E, \Omega)$ from below, we shall state only the following simple result:

THEOREM 1.24. *Let p_1 be the smallest member of E . If $x \geq p_1$ and $0 \leq y \leq (\log x)(\log p_1)^{-1} - 1$, then*

$$S(x, y; E, \Omega) \geq (1/2) p_1^{-y-1} x.$$

To prove this, let $k = [y] + 1$ (so k is the smallest integer greater than y), and observe that the multiples n of p_1^k have the property that $\Omega(n; E) \geq k > y$.

There are just $[xp_1^{-k}]$ of these $n \leq x$, and since $[z] \geq z/2$ for $z \geq 1$, we get the result.

It is clear that Theorem 1.24 is essentially best possible in certain extreme cases (for example, if $E = \{p_1\}$, or if $x = p_1^a$ and $y = a - 1$).

When $E = P$ (the set of all primes), we can take $v = \log_2 x$. Then $\Lambda = O(1)$, and we have the following corollary of Theorems 1.21 and 1.24:

COROLLARY 1.25. *If $x \geq e^e$ and $0 \leq y \leq (\log x)(\log 2)^{-1} - 1$, then*

$$2^{-y-2} x \leq S(x, y; P, \Omega) \leq c_6 2^{-y} x (\log x) (\log_2 x)^{1/2}.$$

Corollary 1.25 should be compared with the Erdős-Sárközy result (1.20) and with the asymptotic formula of Selberg mentioned after Theorem 1.18. When $y < 2 \log_2 x$ (roughly), more precise estimates for $S(x, y; P, \Omega)$ can be obtained from [13] and [14].

In a later paper, we shall show that if p_1 is the smallest member of E and $\varepsilon > 0$ is fixed, then the precise order of magnitude of $S(x, y; E, \Omega)$ is

$$p_1^{-y} x \exp \{(p_1 - 1) E(x)\}$$

when $E(x)$ is sufficiently large and

$$p_1 E(x) \leq y \leq (1 - \varepsilon) (\log x) (\log p_1)^{-1}.$$

This theorem is much more difficult to prove than Theorem 1.21. Its proof depends on Theorem 1.21 and on an extension of Halász's work [4] concerning the local distribution of $\Omega(n; E)$. Theorem 1.21 remains our best upper bound when y is close to $(\log x) (\log p_1)^{-1}$ (cf. Theorem 1.18), and it seems to be the most we can achieve by a fairly simple method.

§2. NOTATION

The symbols a, m, n always represent integers with $a \geq 0, m \geq 0, n > 0$. The letter p always denotes a prime, while $v, w, x, y, z, \alpha, \beta, \delta, \varepsilon, \sigma$ are real numbers. $[x]$ means the largest integer $\leq x$. The notation $\log_2 x$ is defined by (1.5), and the notations $O, O_{\delta, \varepsilon, \dots}, c_i, c_i(\delta, \varepsilon, \dots)$ are explained after Theorem 1.6. If a condition such as " $x \geq c_i(\delta, \varepsilon, \dots)$ " is used as a hypothesis, it is to be understood that $c_i(\delta, \varepsilon, \dots)$ is sufficiently large. We shall occasionally use the notations \ll, \gg to imply constants which are *absolute*. (Thus $A = O(B)$ is equivalent to $A \ll B$.)