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THEOREM 3.20. *Suppose there exist real numbers* $\delta > 0$, $x_0 \geqslant 2$ *such that* (3.14) *holds. If* $x \geqslant c_{29}(\delta)$ *and* $x_0 \leqslant y \leqslant \delta(\log x)(\log_2 x)^{-1}$, *then*

$$S(x, y; E, \omega) \gg x \exp\{-\delta^{-1}(y \log y + \log y + 2)\}.$$

Proof: In the notation of the preceding proof, (3.17) holds, and trivially $\log n_r \leqslant r \log p_r$. If $y \geqslant x_0$, then $p_r > r \geqslant x_0$ and $r = \pi(p_r; E) \geqslant p_r^\delta$, so

$$\log n_r \leqslant \delta^{-1} r \log r \leqslant \delta^{-1}(y+1)(\log y + y^{-1}) \leqslant \delta^{-1}(y \log y + \log y + 2). \tag{3.21}$$

But $\log y \leqslant \log_2 x - \log_3 x$, so $\log n_r < \log x$ if $x \geqslant c_{29}(\delta)$. Hence (3.19) holds, and the result follows from (3.17) and (3.21). Q.E.D.

§4. PROOFS OF THEOREM 1.14 AND RELATED RESULTS

We begin by quoting the following easy result from [13, pp. 689-690]:

LEMMA 4.1. *For* $x \geqslant 1$ *and* $z \geqslant 1$,

$$\sum_{n \leqslant x} z^{\omega(n;E)} \leqslant x \prod_{p \leqslant x,\, p \in E} \{1 + (z-1)p^{-1}\}.$$

To put this in a more convenient form, we prove

LEMMA 4.2. *If* $x \geqslant 1$ *and* $w \geqslant -2$, *then (cf.* (1.2)*)*

$$\prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) \leqslant e^{wE(x)}. \tag{4.3}$$

If $1 \leqslant w \leqslant x$, *then*

$$\prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) = \exp\{w(E(x) - E(w)) + O(w/\log(2w))\}. \tag{4.4}$$

Proof: (4.3) follows immediately from the inequalities

$$0 \leqslant 1 + wp^{-1} \leqslant \exp(wp^{-1}).$$

To get (4.4), we first write

$$\prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) \leqslant \prod_{p \leqslant w} (2wp^{-1}) \cdot \prod_{w < p \leqslant x,\, p \in E} \exp(wp^{-1})$$
$$= \exp\{w(E(x) - E(w)) + \pi(w)\log(2w) - \theta(w)\},$$

where $\pi(w) = \sum_{p \leqslant w} 1$ and $\theta(w) = \sum_{p \leqslant w} \log p$. Since $\pi(t) \ll t/\log(2t)$ for $t \geqslant 1$, we have

$$\theta(w) = \int_1^w (\log t)\, d\pi(t) = \pi(w) \log w - \int_1^w \pi(t)\, t^{-1}\, dt$$
$$= \pi(w) \log w + O\big(w/\log(2w)\big),$$

and it follows that the right-hand side of (4.4) is an upper bound for the left-hand side. On the other hand, since $\log(1+y) = y + O(y^2)$ for $y > 0$, we have

$$\prod_{p \leqslant x,\, p \in E} (1 + wp^{-1}) \geqslant \prod_{w < p \leqslant x,\, p \in E} \exp\{wp^{-1} + O(w^2 p^{-2})\}$$
$$= \exp\{w\big(E(x) - E(w)\big) + O\big(w^2 \sum_{p > w} p^{-2}\big)\}.$$

But

$$\sum_{p > w} p^{-2} = \int_w^{+\infty} t^{-2}\, d\pi(t) < 2 \int_w^{+\infty} t^{-3}\, \pi(t)\, dt \ll \big(w \log(2w)\big)^{-1},$$

and (4.4) follows. Q.E.D.

COROLLARY 4.5. *If* $x \geqslant 1$ *and* $z \geqslant 1$, *then*

$$\sum_{n \leqslant x} z^{\omega(n; E)} \leqslant x\, e^{(z-1) E(x)}. \tag{4.6}$$

If $1 \leqslant z \leqslant x$, *then*

$$\sum_{n \leqslant x} z^{\omega(n; E)}$$
$$\leqslant x \exp\{(z-1)\big(E(x) - E(z)\big) + c_{30} z/\log(2z)\}. \tag{4.7}$$

Note that if $1 \leqslant z < 2$, then (4.7) follows from (4.6).

THEOREM 4.8. *Let* $x \geqslant 1, v > 0, 1 \leqslant \alpha \leqslant x$. *Define* $\Lambda = \Lambda(x, v; E)$ *by* (1.22). *Then*

$$S(x, \alpha v; E, \omega) \leqslant x \exp\{(\alpha - 1 - \alpha \log \alpha) v - \alpha E(\alpha) + c_{31} \Lambda \alpha\}.$$

Proof: Suppose $1 \leqslant z \leqslant x$. Then

$$\sum_{n \leqslant x} z^{\omega(n; E)} \geqslant \sum_{n \leqslant x,\, \omega(n; E) > \alpha v} z^{\omega(n; E)} \geqslant z^{\alpha v} S(x, \alpha v; E, \omega).$$

Combining this result with (4.7), we get

$$S(x, \alpha v; E, \omega) \leqslant x \exp\{(z-1)(v + \Lambda) - zE(z) - \alpha v \log z$$
$$+ c_{32}\, z/\log(2z)\}. \tag{4.9}$$

In practice, we think of v as being a good approximation to $E(x)$, so that Λ is small compared to v. We want to minimize the right-hand side of (4.9) approximately, and for simplicity, we choose z so as to minimize the expression $(z-1)v - \alpha v \log z$, i.e., we take $z = \alpha$. With this value of z, we get the result from (4.9). Q.E.D.

LEMMA 4.10. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Then there is a real number* $\delta(E)$ *such that*

$$E(x) = \gamma(E)\log_2 x + \delta(E) + O_E(1/\log x) \qquad \text{for} \quad x \geqslant 2. \tag{4.11}$$

Proof: Write

$$E(x) = \int_1^x t^{-1}\, d\pi(t; E),$$

integrate by parts, and use (1.7). Q.E.D.

From Theorem 4.8 and Lemma 4.10, we get

COROLLARY 4.12. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Let* $x \geqslant 3$, $2 \leqslant \alpha \leqslant x$. *Then*

$$S\left(x, \alpha\gamma(E)\log_2 x; E, \omega\right)$$

$$\leqslant x \exp\left\{(\alpha - 1 - \alpha\log\alpha)\gamma(E)\log_2 x - \alpha\gamma(E)\log_2\alpha + c_{33}(E)\alpha\right\}.$$

Using (1.8), it is easy to show that Corollary 4.12 actually holds for all $\alpha \geqslant 2$, but it is also clear from (1.8) that

$$S\left(x, \alpha\gamma(E)\log_2 x; E, \omega\right) = 0$$

whenever α is somewhat greater than $(\log x)(\log_2 x)^{-2}$.

The upper bound given in Corollary 4.12 compares favorably with the theorem of Delange (Theorem 1.17 above), and our result is more general and holds for a much wider range of α. Our proof is also much simpler than Delange's. Unfortunately, our lower bound (1.13) is much smaller than the upper bound in Corollary 4.12.

Theorem 1.14 is proved in the same way as Theorem 4.8, but we use (4.6) instead of (4.7), apply Lemma 4.10, and take $z = y\left(\gamma(E)\log_2 x\right)^{-1}$.

We conclude this section by generalizing the Erdös-Nicolas result (1.10).

THEOREM 4.13. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Let* $\varepsilon > 0$, *and suppose that* $x \geqslant c_{34}(E, \varepsilon)$ *and*

$$(\log_2 x)^2 (\log x)^{-1} \leqslant \alpha \leqslant 1 + \{1 + \log \gamma(E) - \varepsilon\} (\log_2 x)^{-1}.$$

Then

$$x^{1-\alpha} \exp\left\{-c_{35}(E) \frac{\log x}{\log_2 x}\right\} \leqslant S\left(x, \alpha (\log x) (\log_2 x)^{-1}; E, \omega\right)$$
$$\leqslant x^{1-\alpha} \exp\left\{\frac{2\alpha (\log x) \log_3 x}{\log_2 x} + c_{36}(E) \frac{\log x}{\log_2 x}\right\}.$$

This can be obtained from Theorems 1.11 and 1.14 (take

$$y = \alpha (\log x) (\log_2 x)^{-1}$$

and use the inequalities

$$\log_2 y \leqslant \log_3 x, y \geqslant \log_2 x \geqslant \gamma(E) \log_2 x).$$

Theorem 4.13 should be compared with Theorem 1.6.

§5. PROOFS OF THEOREM 1.21 AND RELATED RESULTS

In estimating $S(x, y; E, \Omega)$ (defined by (1.1)), we do not need any assumption such as (1.7). Hence we emphasize that throughout the remainder of this paper, E is merely assumed to be any nonempty set of primes. (We shall sometimes assume explicitly that E has at least two members.) The smallest member of E will always be denoted by p_1 (and the smallest member of $E - \{p_1\}$, if it exists, by p_2). When x and v are positive real numbers, the function $\Lambda = \Lambda(x, v; E)$ is always defined by (1.22).

The subsequent work depends heavily on the following elementary lemma [13, p. 690]:

LEMMA 5.1. *If* $x > 0$ *and* $1 \leqslant z < p_1$, *then*

$$\sum_{n \leqslant x} z^{\Omega(n; E)} < p_1 (p_1 - z)^{-1} x\, e^{(z-1) E(x) + 4z}.$$