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**THEOREM 4.13.** *Suppose that there exists a real number  $\gamma(E) > 0$  such that (1.7) holds. Let  $\varepsilon > 0$ , and suppose that  $x \geq c_{34}(E, \varepsilon)$  and  $(\log_2 x)^2 (\log x)^{-1} \leq \alpha \leq 1 + \{1 + \log \gamma(E) - \varepsilon\} (\log_2 x)^{-1}$ .*

*Then*

$$\begin{aligned} x^{1-\alpha} \exp \left\{ -c_{35}(E) \frac{\log x}{\log_2 x} \right\} &\leq S(x, \alpha (\log x) (\log_2 x)^{-1}; E, \omega) \\ &\leq x^{1-\alpha} \exp \left\{ \frac{2\alpha (\log x) \log_3 x}{\log_2 x} + c_{36}(E) \frac{\log x}{\log_2 x} \right\}. \end{aligned}$$

This can be obtained from Theorems 1.11 and 1.14 (take

$$y = \alpha (\log x) (\log_2 x)^{-1}$$

and use the inequalities

$$\log_2 y \leq \log_3 x, \quad y \geq \log_2 x \geq \gamma(E) \log_2 x.$$

Theorem 4.13 should be compared with Theorem 1.6.

## §5. PROOFS OF THEOREM 1.21 AND RELATED RESULTS

In estimating  $S(x, y; E, \Omega)$  (defined by (1.1)), we do not need any assumption such as (1.7). Hence we emphasize that throughout the remainder of this paper,  $E$  is merely assumed to be any nonempty set of primes. (We shall sometimes assume explicitly that  $E$  has at least two members.) The smallest member of  $E$  will always be denoted by  $p_1$  (and the smallest member of  $E - \{p_1\}$ , if it exists, by  $p_2$ ). When  $x$  and  $v$  are positive real numbers, the function  $\Lambda = \Lambda(x, v; E)$  is always defined by (1.22).

The subsequent work depends heavily on the following elementary lemma [13, p. 690]:

**LEMMA 5.1.** *If  $x > 0$  and  $1 \leq z < p_1$ , then*

$$\sum_{n \leq x} z^{\Omega(n; E)} < p_1 (p_1 - z)^{-1} x e^{(z-1)E(x) + 4z}.$$

For the special case  $E = P$ , there is a recent paper of DeKoninck and Hensley [1] giving various estimates for  $\sum_{n \leq x}^* z^{\Omega(n)}$ , where  $z$  is complex and \* indicates that the prime factors of  $n$  are restricted to lie in a certain range. DeKoninck and Hensley get sharp results, but their work is rather complicated and does not seem applicable to the problems discussed here.

If  $y$  is real and  $z \geq 1$ , then

$$\begin{aligned} \sum_{n \leq x} z^{\Omega(n; E)} &\geq \sum_{n \leq x, \Omega(n; E) \geq y} z^{\Omega(n; E)} \\ &\geq z^y \text{card} \{n \leq x : \Omega(n; E) \geq y\}. \end{aligned}$$

Hence Lemma 5.1 immediately yields

LEMMA 5.2. *If  $x > 0$ ,  $y$  is real, and  $1 \leq z < p_1$ , then*

$$\begin{aligned} &\text{card} \{n \leq x : \Omega(n; E) \geq y\} \\ &< p_1 (p_1 - z)^{-1} x \exp \{(z-1) E(x) - y \log z + 4z\}. \end{aligned}$$

LEMMA 5.3. *Let  $x > 0$ ,  $0 < v \leq y < p_1 v$ . Then*

$$\begin{aligned} &\text{card} \{n \leq x : \Omega(n; E) \geq y\} \\ &< c_{37} (p_1) (p_1 - y/v)^{-1} x \exp \{y - v - y \log (y/v) + p_1 \Lambda\}. \end{aligned}$$

*Proof:* In Lemma 5.2, use the inequality  $E(x) \leq v + \Lambda$  and take  $z = y/v$  to get an approximate minimum. Q.E.D.

We observe in passing that Lemma 5.2 can also be used when  $y \geq p_1 v$ . In order to get a reasonably good result in this case by the same method, one needs to minimize the function

$$g(z) = (z-1)v - y \log z - \log(p_1 - z)$$

on the interval  $1 \leq z < p_1$ . Assuming that  $y$  is rather large, one can see with some computation that  $g(z)$  is approximately minimized when

$$z = p_1 (1 - (2y)^{-1}),$$

and this  $z$  satisfies  $1 \leq z < p_1$  whenever  $y \geq 1$ . With this value of  $z$ , Lemma 5.2 yields

$$\text{card} \{n \leq x : \Omega(n; E) \geq y\} \leq c_{38} (p_1) y p_1^{-y} x e^{(p_1-1)v + p_1 \Lambda} \quad (5.4)$$

for  $x > 0$ ,  $y \geq 1$ . When  $E$  is the set of all primes and  $x \geq 3$ , we can take  $v = \log_2 x$ ,  $\Lambda = O(1)$ . Thus (5.4) is already sharper and more general

than (1.20) (which is due to Erdős and Sárközy [3]). However, Theorem 1.18 shows that it may be of interest to take  $y$  as large as  $(\log x) (\log p_1)^{-1}$ , and we shall now prove that when  $y$  is relatively large, the factor  $y$  on the right-hand side of (5.4) can be replaced by a much smaller quantity.

LEMMA 5.5. Write  $F = E - \{p_1\}$  (if  $F$  is empty, we define  $\Omega(n; F) = 0$  for all  $n$ ). Let  $x > 0, y \geq 0$ , and let  $k = [y] + 1$ . For integers  $a$  with  $0 \leq a \leq k$ , define

$$C_a = \{m \leq xp_1^{-a} : p_1 \nmid m \text{ and } \Omega(m; F) \geq k - a\}.$$

Then

$$S(x, y; E, \Omega) = [xp_1^{-k}] + \sum_{a=0}^{k-1} \text{card } C_a.$$

*Proof:* For  $0 \leq a \leq k$ , define

$$B_a = \{n \leq x : p_1^a \parallel n \text{ and } \Omega(np_1^{-a}; F) \geq k - a\}$$

(recall that  $p_1^a \parallel n$  means  $p_1^a \mid n$  and  $p_1^{a+1} \nmid n$ ). It is easy to see that

$$\{n \leq x : \Omega(n; E) > y\} = \{n \leq x : p_1^k \mid n\} \cup \bigcup_{a=0}^{k-1} B_a.$$

Since the sets  $\{n \leq x : p_1^k \mid n\}, B_0, B_1, \dots, B_{k-1}$  are disjoint, we have

$$S(x, y; E, \Omega) = \text{card } \{n \leq x : p_1^k \mid n\} + \sum_{a=0}^{k-1} \text{card } B_a.$$

But the mapping  $n \mapsto np_1^{-a}$  establishes a one-to-one correspondence between  $B_a$  and  $C_a$ , so the result follows. Q.E.D.

*Proof of Theorem 1.21:* If  $E = \{p_1\}$ , then by Lemma 5.5,

$$S(x, y; E, \Omega) \leq xp_1^{-y},$$

and (1.23) follows. Thus we may assume that  $F = E - \{p_1\}$  is not empty. Let  $p_2$  be the smallest member of  $F$ , and let  $k = [y] + 1$ . By Lemma 5.5,

$$S(x, y; E, \Omega) = [xp_1^{-k}] + \sum_{a=1}^k \text{card } C_{k-a}. \quad (5.6)$$

To estimate

$$\text{card } C_{k-a} = \text{card } \{m \leq xp_1^{a-k} : p_1 \nmid m \quad \text{and} \quad \Omega(m; F) \geq a\}$$

from above, we apply Lemma 5.2 (with  $E$  replaced by  $F$  and  $p_1$  by  $p_2$ ). Since

$$F(xp_1^{a-k}) \leq F(x) \leq E(x) \leq v + \Lambda,$$

we obtain

$$\begin{aligned} \text{card } C_{k-a} &< p_2 (p_2 - z)^{-1} xp_1^{a-k} \exp \{(z-1)(v+\Lambda) - a \log z + 4z\} \\ &= H(a, z), \end{aligned} \tag{5.7}$$

say, and this holds for each integer  $a$  ( $1 \leq a \leq k$ ) and each real  $z$  with  $1 \leq z < p_2$ . In applying (5.7), we are free to choose  $z$  to depend on  $a$ . Write  $Q = \max \{k, p_1 v\}$ , and for each  $a$  ( $1 \leq a \leq Q$ ), let  $z_a$  be any real number satisfying  $1 \leq z_a < p_2$ . Then by (5.6) and (5.7),

$$\begin{aligned} S(x, y; E, \Omega) &\leq xp_1^{-k} + \sum_{a=1}^k H(a, z_a) \\ &\leq xp_1^{-k} + \sum_{1 \leq a \leq v} H(a, z_a) + \sum_{v < a \leq p_1 v} H(a, z_a) \\ &\quad + \sum_{p_1 v < a \leq Q} H(a, z_a). \end{aligned} \tag{5.8}$$

For  $1 \leq a \leq v$ , take  $z_a = 1$ . With this choice, we have

$$\begin{aligned} \sum_{1 \leq a \leq v} H(a, z_a) &\ll xp_1^{-k} \sum_{1 \leq a \leq v} p_1^a \ll xp_1^{-y+v} \\ &\ll xp_1^{-y} e^{(p_1-1)v}. \end{aligned} \tag{5.9}$$

For  $v < a \leq p_1 v$ , the quantity  $(z-1)v - a \log z$  in (5.7) is minimized by taking  $z = a/v = z_a$ . With this choice of  $z_a$ , we have  $1 < z_a \leq p_1$  and

$$p_2 (p_2 - z_a)^{-1} \leq p_2 (p_2 - p_1)^{-1} \leq 1 + p_1,$$

so

$$H(a, z_a) \leq c_{39} (p_1) xp_1^{a-k} e^{(p_1-1)\Lambda} (v^a e^{-v} / a^a e^{-a}).$$

By Stirling's formula,  $a^a e^{-a} \gg a! a^{-1/2}$ , so we get

$$\begin{aligned} \sum_{v < a \leq p_1 v} H(a, z_a) &\leq c_{40} (p_1) xp_1^{-y} v^{1/2} e^{-v+p_1\Lambda} \sum_{v < a \leq p_1 v} \frac{(p_1 v)^a}{a!} \\ &\leq c_{40} (p_1) xp_1^{-y} v^{1/2} e^{(p_1-1)v+p_1\Lambda}. \end{aligned} \tag{5.10}$$

For  $p_1 v < a \leq Q$ , we let all the numbers  $z_a$  have the same value  $p_1 (1 + \theta)$ , where  $\theta$  is a real number about which we assume only that  $0 < \theta < p_2 p_1^{-1} - 1$  (the last inequality being needed in order to have  $z_a < p_2$ ). With this choice of  $z_a$ , (5.7) yields

$$\begin{aligned} & \sum_{p_1 v < a \leq Q} H(a, z_a) \\ & \leq p_2 \{p_2 - p_1 (1 + \theta)\}^{-1} x p_1^{-k} \exp \{(p_1 - 1 + p_1 \theta) (v + \Lambda) + 4p_1 (1 + \theta)\} \\ & \quad \times \sum_{p_1 v < a \leq Q} (1 + \theta)^{-a}. \end{aligned} \quad (5.11)$$

The last sum on the right does not exceed

$$\sum_{a > p_1 v} (1 + \theta)^{-a} < (1 + \theta) \theta^{-1} (1 + \theta)^{-p_1 v}. \quad (5.12)$$

After combining this estimate with (5.11), we would like to minimize the contribution of the essential terms  $e^{p_1 \theta v} \theta^{-1} (1 + \theta)^{-p_1 v}$ . Since

$$\log(1 + \theta) \geq \theta - \theta^2/2 \quad \text{for } \theta \geq 0, \quad (5.13)$$

we have

$$p_1 \theta v - \log \theta - p_1 v \log(1 + \theta) \leq -\log \theta + p_1 v \theta^2/2,$$

and here the right-hand side would be minimized by taking  $\theta$  to be  $(p_1 v)^{-1/2}$ . However, we must also choose  $\theta < p_2 p_1^{-1} - 1$  (so that  $z_a < p_2$ ). If we take

$$\theta = (2p_1 v^{1/2})^{-1}, \quad (5.14)$$

then because of our assumption that  $v \geq 1$ , we have

$$\theta \leq (2p_1)^{-1} < p_2 p_1^{-1} - 1.$$

Combining (5.11), (5.12), (5.13), and (5.14), and observing that

$$\begin{aligned} p_2 \{p_2 - p_1 (1 + \theta)\}^{-1} & \leq p_2 (p_2 - p_1 - 1/2)^{-1} \\ & = 1 + (p_1 + 1/2) (p_2 - p_1 - 1/2)^{-1} \leq c_{41} (p_1), \end{aligned}$$

we obtain finally

$$\sum_{p_1 v < a \leq Q} H(a, z_a) \leq c_{42} (p_1) x p_1^{-y} v^{1/2} e^{(p_1 - 1)v + p_1 \Lambda}. \quad (5.15)$$

The theorem now follows from (5.8), (5.9), (5.10), and (5.15). Q.E.D.

Since

$$E(x) \leq \sum_{p \leq x} p^{-1} = \log_2 x + O(1) \quad \text{for } x \geq 2,$$

one would always want to choose  $v \leq \log_2 x$ . Thus (1.23) is superior to (5.4) whenever  $y \geq (\log_2 x)^{1/2}$ . Furthermore, consideration of derivatives shows that

$$y - v - y \log(y/v) \leq (p_1 - 1)v - y \log p_1 \quad \text{for } 0 < v \leq y \leq p_1 v,$$

and hence Lemma 5.3 is superior to Theorem 1.21 whenever

$$1 \leq v \leq y \leq p_1 v - v^{1/2}.$$

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