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Artikel: THE REPRESENTATION THEORY OF SL(2, R), A NON-INFINITESIMAL APPROACH

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2.2. The principal series

Let G be a connected noncompact real semisimple Lie group with finite center. Let $G = KAN$ be an Iwasawa decomposition. For $g \in G$ write $g = u(g)\exp(H(g))n(g)$, where $u(g) \in K$, $H(g) \in \mathfrak{a}$ (the Lie algebra of A) and $n(g) \in N$. Let $\rho \in \mathfrak{a}^*$ be half the sum of the positive roots. Let M be the centralizer of A in K. For $\xi \in \hat{M}$, $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ the *principal series* representation $\pi_{\xi,\lambda}$ of G is obtained by inducing the (not necessarily unitary) finite-dimensional irreducible representation $man \to e^{\lambda(\log a)}\xi(m)$ of the subgroup MAN. In the so-called compact picture we have the following realization of $\pi_{\xi,\lambda}$ (cf. Wallach [45, §8.3]):

$$(\pi_{\xi,\lambda}(g)f)(k) = e^{-(\rho+\lambda)(H(g^{-1}k))}\, f(u(g^{-1}k)), \quad f \in L^2_\xi(K, \mathscr{H}(\xi)), \quad k \in K, g \in G. \tag{2.2}$$

Here the Hilbert space $L^2_\xi(K, \mathscr{H}(\xi))$ consists of all $\mathscr{H}(\xi)$-valued L^2-functions f on K such that $f(km) = \xi(m^{-1})f(k)$, $k \in K$, $m \in M$. The representation $\pi_{\xi,\lambda}$ is a K-unitary Hilbert representation. It is unitary if $\lambda \in i\mathfrak{a}^*$. By Frobenius reciprocity, $\pi_{\xi,\lambda}$ is K-finite and $\pi_{\xi,\lambda}$ is K-multiplicity free if each $\delta \in \hat{K}$ is M-multiplicity free.

Let us now specialize the above results to $G = SL(2, \mathbf{R})$. It is convenient to work with the group $G = SU(1, 1)$, isomorphic to $SL(2, \mathbf{R})$:

$$G := \left\{ g_{\alpha,\beta} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix};\ \alpha, \beta \in \mathbf{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}. \tag{2.3}$$

Let

$$K := \left\{ u_\theta = \begin{pmatrix} e^{\frac{1}{2}i\theta} & 0 \\ 0 & e^{-\frac{1}{2}i\theta} \end{pmatrix};\ 0 \leqslant \theta < 4\pi \right\}, \tag{2.4}$$

$$A := \left\{ a_t = \begin{pmatrix} ch\frac{1}{2}t & sh\frac{1}{2}t \\ sh\frac{1}{2}t & ch\frac{1}{2}t \end{pmatrix} = \exp t \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix};\ t \in \mathbf{R} \right\}, \tag{2.5}$$

$$N := \left\{ n_z = \begin{pmatrix} 1+\frac{1}{2}iz & -\frac{1}{2}iz \\ \frac{1}{2}iz & 1-\frac{1}{2}iz \end{pmatrix};\ z \in \mathbf{R} \right\}. \tag{2.6}$$

Then $G = KAN$ is an Iwasawa decomposition for $G = SU(1, 1)$, $\rho(\log a_t) = \frac{1}{2}t$ and $M = \{u_0, u_{2\pi}\}$. $\hat{M}$ consists of the two one-dimensional representations

(2.7) $$u_\theta \to e^{i\xi\theta}, \quad u_\theta \in M, \xi = 0 \text{ or } \tfrac{1}{2}.$$

Let $L^2_\xi(K)$ consist of all $f \in L^2(K)$ such that $f(u_{\psi+2\pi}) = f(u_\psi)$ or $-f(u_\psi)$ according to whether $\xi = 0$ or $\frac{1}{2}$, respectively.

Now, by using explicit expressions for the factors in the Iwasawa decomposition of $g^{-1}_{\alpha,\beta}\, u_\psi$ (cf. TAKAHASHI [39, §1]) we can write (2.2) in the case $G = SU(1, 1)$ as follows:

(2.8) $$\begin{gathered}(\pi_{\xi,\lambda}(g_{\alpha,\beta})f)(u_\psi) := |\bar{\alpha} e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}|^{-2\lambda-1} f(u_{\psi'}),\\ \psi' := 2\arg(\bar{\alpha} e^{\frac{1}{2}i\psi} - \beta e^{-\frac{1}{2}i\psi}), \quad g_{\alpha,\beta} \in G,\ u_\psi \in K,\ f \in L^2_\xi(K),\\ \xi = 0 \text{ or } \tfrac{1}{2}, \lambda \in \mathbf{C}.\end{gathered}$$

On putting $g_{\alpha,\beta} := u_\theta \in K$ we get

(2.9) $$(\pi_{\xi,\lambda}(u_\theta)f)(u_\psi) = f(u_{\psi-\theta}), \quad f \in L^2_\xi(K),\ u_\theta, u_\psi \in K,$$

which again shows that $\pi_{\xi,\lambda}$ is K-unitary. $\hat{K}$ consists of the representations

(2.10) $$\delta_n(u_\theta) := e^{in\theta}, \quad u_\theta \in K,$$

where n runs through the set $\frac{1}{2}\mathbf{Z}$, i.e., $2n \in \mathbf{Z}$. An orthogonal basis for $L^2_\xi(K)$ is given by the functions

(2.11) $$\phi_n(u_\psi) := e^{-in\psi}, \quad u_\psi \in K,$$

where n runs through the set $\mathbf{Z} + \xi := \{m+\xi \mid m \in \mathbf{Z}\}$. Then

(2.12) $$\pi_{\xi,\lambda}(u_\theta)\phi_n = \delta_n(u_\theta)\phi_n, \quad u_\theta \in K, n \in \mathbf{Z} + \xi.$$

Thus $\pi_{\xi,\lambda}$ is K-multiplicity free,

(2.13) $$\mathscr{M}(\pi_{\xi,\lambda}) = \{\delta_n \in \hat{K} \mid n \in \mathbf{Z} + \xi\},$$

the ϕ_n's form a K-basis for $L^2_\xi(K)$ and the canonical matrix elements of $\pi_{\xi,\lambda}$ are

(2.14) $$\pi_{\xi,\lambda,m,n}(g) = (\pi_{\xi,\lambda}(g)\phi_n, \phi_m), \quad g \in G,\ m, n \in \mathbf{Z} + \xi.$$

Because of the Cartan decomposition $G = KAK$, $\pi_{\xi,\lambda,m,n}$ is completely determined by its restriction to A. It follows from (2.8) and (2.11) that

$$\begin{aligned}(\pi_{\xi,\lambda}(a_t)\phi_n)(u_\psi) &= |\, ch\tfrac{1}{2}t\, e^{\frac{1}{2}i\psi} - sh\tfrac{1}{2}t\, e^{-\frac{1}{2}i\psi}|^{-2\lambda+2n-1}\\ &\quad \cdot (ch\tfrac{1}{2}t\, e^{\frac{1}{2}i\psi} - sh\tfrac{1}{2}t\, e^{-\frac{1}{2}i\psi})^{-2n}.\end{aligned}$$

Hence

(2.15)
$$\pi_{\xi,\lambda,m,n}(a_t) = (ch\tfrac{1}{2}t)^{-2\lambda-1}$$
$$\cdot \frac{1}{4\pi}\int_0^{4\pi} (1-th\tfrac{1}{2}t\, e^{i\psi})^{-\lambda+n-1/2}(1-th\tfrac{1}{2}t\, e^{-i\psi})^{-\lambda-n-1/2} e^{i(m-n)\psi} d\psi\,,$$
$$t \in \mathbf{R},\ m,n \in \mathbf{Z} + \xi\,.$$

The following symmatry is evident from (2.15):

(2.16)
$$\pi_{\xi,\lambda,-m,-n}(a_t) = \pi_{\xi,\lambda,m,n}(a_t)\,.$$

2.3. Calculation of the canonical matrix elements

Let us calculate the integral (2.15). In view of (2.16) we can suppose $m \geqslant n$. The binomial expansion

(2.17)
$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad |z| < 1, a \in \mathbf{C}\,,$$

where

(2.18)
$$(a)_k := a(a+1)\ldots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}\,,$$

can be substituted for the first two factors in the integrand of (2.15). Now interchange the order of summation and integration and perform the integration in each term. Then we obtain $(m \geqslant n)$

(2.19)
$$\pi_{\xi,\lambda,m,n}(a_t) = \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!} (sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{n-m-2\lambda-1}$$
$$\cdot\, {}_2F_1(\lambda+m+\tfrac{1}{2}, \lambda-n+\tfrac{1}{2}; m-n+1; (th\tfrac{1}{2}t)^2)\,,$$

where the ${}_2F_1$ denotes a *hypergeometric series*, defined by

(2.20)
$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!} z^k, \quad |z| < 1, a,b,c \in \mathbf{C}\,,$$

cf. [10, Vol. I, Ch. 2].

The expression (2.20) is clearly symmetric in a and b. As a function of z, the ${}_2F_1$ has an analytic continuation to a one-valued function on $\mathbf{C}\backslash[1, \infty)$. Application of the transformation formulas