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Hence

$$\pi_{\xi,\lambda,m,n}(a_t) = (ch\tfrac{1}{2}t)^{-2\lambda-1} \tag{2.15}$$

$$\cdot \frac{1}{4\pi}\int_0^{4\pi} (1-th\tfrac{1}{2}t\, e^{i\psi})^{-\lambda+n-1/2}(1-th\tfrac{1}{2}t\, e^{-i\psi})^{-\lambda-n-1/2} e^{i(m-n)\psi}d\psi\,,$$

$$t \in \mathbf{R},\ m,n \in \mathbf{Z} + \xi\,.$$

The following symmatry is evident from (2.15):

$$\pi_{\xi,\lambda,-m,-n}(a_t) = \pi_{\xi,\lambda,m,n}(a_t)\,. \tag{2.16}$$

2.3. Calculation of the canonical matrix elements

Let us calculate the integral (2.15). In view of (2.16) we can suppose $m \geqslant n$. The binomial expansion

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad |z| < 1, a \in \mathbf{C}\,, \tag{2.17}$$

where

$$(a)_k := a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}\,, \tag{2.18}$$

can be substituted for the first two factors in the integrand of (2.15). Now interchange the order of summation and integration and perform the integration in each term. Then we obtain $(m \geqslant n)$

$$\pi_{\xi,\lambda,m,n}(a_t) = \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{n-m-2\lambda-1} \tag{2.19}$$

$$\cdot\, {}_2F_1(\lambda+m+\tfrac{1}{2}, \lambda-n+\tfrac{1}{2}; m-n+1; (th\tfrac{1}{2}t)^2)\,,$$

where the ${}_2F_1$ denotes a *hypergeometric series*, defined by

$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!} z^k, \quad |z| < 1, a,b,c \in \mathbf{C}\,, \tag{2.20}$$

cf. [10, Vol. I, Ch. 2].

The expression (2.20) is clearly symmetric in a and b. As a function of z, the ${}_2F_1$ has an analytic continuation to a one-valued function on $\mathbf{C}\backslash[1, \infty)$. Application of the transformation formulas

$$(2.21) \qquad {}_2F_1(a, b; c; z) = (1-z)^{-b}{}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

(cf. [10, Vol. I, §2.1 (22)]) to (2.19) yields ($m \geq n$):

$$(2.22) \qquad \pi_{\xi, \lambda, m, n}(a_t)$$

$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{-m-n}{}_2F_1(\lambda-n+\tfrac{1}{2}, -\lambda-n+\tfrac{1}{2}; m-n+1; -(sh\tfrac{1}{2}t)^2)$$

$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)!}(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{m+n}{}_2F_1(\lambda+m+\tfrac{1}{2}, -\lambda+m+\tfrac{1}{2}; m-n+1; -(sh\tfrac{1}{2}t)^2).$$

It is more elegant to express the hypergeometric functions in (2.22) in terms of *Jacobi functions* $\phi_\mu^{(\alpha,\beta)}$ ($\mu, \alpha, \beta \in \mathbf{C}$, $\alpha \notin \{-1, -2, \ldots\}$), which are defined on $\mathbf{R}$ by

$$(2.23) \qquad \phi_\mu^{(\alpha,\beta)}(t) := {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\mu), \tfrac{1}{2}(\alpha+\beta+1-i\mu); \alpha+1; -(sh t)^2)$$

(cf. KOORNWINDER [36, §2]). Clearly,

$$(2.24) \qquad \phi_\mu^{(\alpha,\beta)}(0) = 1,$$

$$(2.25) \qquad \phi_\mu^{(\alpha,\beta)}(t) = \phi_\mu^{(\alpha,\beta)}(-t),$$

$$(2.26) \qquad \phi_\mu^{(\alpha,\beta)}(t) = \phi_{-\mu}^{(\alpha,\beta)}(t).$$

The function $\phi_\mu^{(\alpha,\beta)}$ satisfies the differential equation

$$(2.27) \qquad (\Delta_{\alpha,\beta}(t))^{-1}\frac{d}{dt}\left(\Delta_{\alpha,\beta}(t)\frac{du(t)}{dt}\right) = -(\mu^2+(\alpha+\beta+1)^2)u(t),$$

where

$$\Delta_{\alpha,\beta}(t) := (sh t)^{2\alpha+1}(ch t)^{2\beta+1},$$

and $u := \phi_\mu^{\alpha,\beta}$ is the unique solution of (2.27) which is regular at $t = 0$ and satisfies $u(0) = 1$. For fixed $\alpha > -1$, $\beta \in \mathbf{R}$, Jacobi functions $\phi_\mu^{(\alpha,\beta)}$ form a continuous orthogonal system with respect to the measure $\Delta_{\alpha,\beta}(t)dt$, $t > 0$.

Substitution of (2.23) and (2.22) yields $(m \geqslant n)$:

(2.28)
$$\pi_{\xi,\lambda,m,n}(a_t)$$
$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)\,!}\,(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{-m-n}\phi_{2i\lambda}^{(m-n,\,-m-n)}(\tfrac{1}{2}t)$$
$$= \frac{(\lambda+n+\frac{1}{2})_{m-n}}{(m-n)\,!}\,(sh\tfrac{1}{2}t)^{m-n}(ch\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(m-n,\,m+n)}(\tfrac{1}{2}t)\,.$$

Application of (2.16) gives a similar result in the case $m < n$. Finally we conclude:

THEOREM 2.1. *The canonical matrix elements* $\pi_{\xi,\lambda,m,n}(a_t)$ $(\lambda \in \mathbf{C};\ \xi = 0$ or $\frac{1}{2};\ m, n \in \mathbf{Z} + \xi;\ t \in \mathbf{R})$ *of* $SU(1, 1)$ *can be expressed in terms of Jacobi functions by*

(2.29)
$$\pi_{\xi,\lambda,m,n}(a_t) = \frac{c_{\xi,\lambda,m,n}}{(|m-n|)\,!}\,(sh\tfrac{1}{2}t)^{|m-n|}\,(ch\tfrac{1}{2}t)^{m+n}\phi_{2i\lambda}^{(|m-n|,\,m+n)}(\tfrac{1}{2}t)\,,$$

where

(2.30)
$$c_{\xi,\lambda,m,n} := \begin{cases} (\lambda+n+\frac{1}{2})_{m-n} & \text{if } m \geqslant n\,,\\ (\lambda-n+\frac{1}{2})_{n-m} & \text{if } n \geqslant m\,. \end{cases}$$

In view of (2.24), formulas (2.29) and (2.30) describe the asymptotics of $\pi_{\xi,\lambda,m,n}$ near $t = 0$.

2.4. NOTES

2.4.1. The principal series of representations was first written down for $SL(2, \mathbf{R})$ by BARGMANN [2], for $SL(2, \mathbf{C})$ by GELFAND & NAIMARK [18], and for a general noncompact semisimple Lie group by HARISH-CHANDRA [21, §12].

2.4.2. BARGMANN [2, §10] already obtained explicit expressions in terms of hypergeometric functions for the canonical matrix elements of the irreducible unitary representations of $SL(2, \mathbf{R})$. He solved the differential equation satisfied by these matrix elements, which is obtained from the Casimir operator. VILENKIN [43, Ch. VI, §3] gives a derivation of these expressions which is similar to our derivation in §2.4, starting from the integral representation (2.15).

2.4.3. It follows from the present paper that the spherical functions for $SL(2, \mathbf{R})$ can be expressed as Jacobi functions of order $(\alpha, \beta) = (0, 0)$. More generally, the spherical functions on any noncompact real semisimple Lie group