Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 28 (1982)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE REPRESENTATION THEORY OF SL(2, R), A NON-INFINITESIMAL APPROACH

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Kapitel: 3. The irreducible subquotient representations of the principal series

DOI: https://doi.org/10.5169/seals-52233

of rank 1 (i.e., $\dim(A) = 1$) can be written as Jacobi functions of certain order (cf. HARISH-CHANDRA [23, §13]). This motivated FLENSTED-JENSEN [14] to study harmonic analysis for Jacobi function expansions of quite general order (α, β), $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. This research was continued in several papers by Flensted-Jensen and the author.

3. THE IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

3.1. SUBQUOTIENT REPRESENTATIONS

We start with the definition and some general properties and next derive an irreducibility criterium (Theorem 3.2) and a decomposition theorem 3.3.

Let G be a lcsc. group and let τ be a Hilbert representation of G. Let $\mathscr{H}_0$ be a closed subspace of $\mathscr{H}(\tau)$ and let P_0 be the orthogonal projection from $\mathscr{H}(\tau)$ onto $\mathscr{H}_0$. Define

(3.1) $$\tau_0(g)v := P_0\tau(g)v\,, \quad g \in G, v \in \mathscr{H}_0\,.$$

Then $\tau(g) \in \mathscr{L}(\mathscr{H}_0)$ for each $g \in G$, $\tau_0(e) = id.$, and $g \to \tau_0(g)v : G \to \mathscr{H}_0$ is continuous for each $v \in \mathscr{H}_0$. If also

(3.2) $$\tau_0(g_1g_2) = \tau_0(g_1)\tau_0(g_2)\,, \quad g_1, g_2 \in G\,,$$

then τ_0 is a Hilbert representation of G on $\mathscr{H}_0$ and it is called a *subquotient representation* of τ. Formula (3.2) is clearly valid if $\mathscr{H}_0$ is an *invariant subspace* of $\mathscr{H}(\tau)$, i.e., if $\tau(g)v \in \mathscr{H}_0$ for all $g \in G$, $v \in \mathscr{H}_0$. In that case, τ_0 is called a *subrepresentation* of τ.

LEMMA 3.1. *Let $\mathscr{H}_0$ be a closed subspace of $\mathscr{H}(\tau)$, let $\mathscr{H}_2$ be the closed G-invariant subspace of $\mathscr{H}(\tau)$ which is generated by $\mathscr{H}_0$ and let $\mathscr{H}_1 := \mathscr{H}_2 \cap \mathscr{H}_0^{\perp}$. Then τ_0 is a subquotient representation if and only if $\mathscr{H}_1$ is G-invariant.*

Proof. Let P_0 and P_1 denote the orthogonal projections on $\mathscr{H}_0$ and $\mathscr{H}_1$, respectively. It follows from (3.1) that

$$\tau_0(g_1 g_2)v - \tau_0(g_1)\tau_0(g_2)v$$
$$= P_0\tau(g_1)P_1\tau(g_2)v\,, \quad g_1, g_2 \in G, v \in \mathscr{H}_0\,.$$

$\mathscr{H}_1$ is the closed linear span of all elements $P_1\tau(g_2)v$, $g_2 \in G$, $v \in \mathscr{H}_0$. So (3.2) holds iff $P_0\tau(g_1)w = 0$ for all $g_1 \in G$, $w \in \mathscr{H}_1$. □

Let K be a compact subgroup of G and suppose that τ is K-unitary. Let τ_0 be a subquotient representation of τ on $\mathscr{H}_0$ and let $\mathscr{H}_1$ and $\mathscr{H}_2$ be as in Lemma 3.1. Then $\mathscr{H}_2$ and $\mathscr{H}_1$ are G-invariant subspaces, so $\mathscr{H}_0 = \mathscr{H}_2 \cap \mathscr{H}_1^{\perp}$ is K-invariant. It follows that τ_0 is K-unitary and that $\tau_0(k)v = \tau(k)v$, $k \in K$, $v \in \mathscr{H}_0$. If K is compact abelian and if τ is K-multiplicity free then τ_0 is also K-multiplicity free, $\mathscr{M}(\tau_0) \subset \mathscr{M}(\tau)$ and $\tau_{0,\gamma,\delta}(g) = \tau_{\gamma,\delta}(g)$ for $\gamma, \delta \in \mathscr{M}(\tau_0)$, $g \in G$.

Let again K be a compact abelian subgroup of G and τ a K-multiplicity free Hilbert representation of G. Let $\mathscr{H}_0$ be a K-invariant closed subspace of $\mathscr{H}(\tau)$. Then, by Lemma 3.1, τ_0 defined by (3.1) is a subquotient representation if and only if we can partition the K-basis for $\mathscr{H}(\tau)$ into three parts, the first part providing a basis for $\mathscr{H}_0$, such that, for each $g \in G$, the corresponding 3×3 block matrix of $(\tau_{\gamma\delta}(g))$ takes the form

$$\begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}. \tag{3.3}$$

THEOREM 3.2. *Let K be a compact abelian subgroup of the lcsc. group G and let τ be a K-multiplicity free Hilbert representation of G. Let τ_0 be a subquotient representation of τ. Then the following three statements are equivalent:*

(a) *τ_0 is irreducible.*

(b) *For some $\delta \in \mathscr{M}(\tau_0)$ we have $\tau_{\gamma\delta} \neq 0 \neq \tau_{\delta\gamma}$ for all $\gamma \in \mathscr{M}(\tau_0)$.*

(c) *For all $\gamma, \delta \in \mathscr{M}(\tau_0)$ we have $\tau_{\gamma\delta} \neq 0$.*

Proof. First note: if $v \in \mathscr{H}(\tau_0)$ and $(v, \phi_\gamma) \neq 0$ for some $\gamma \in \mathscr{M}(\tau_0)$ then ϕ_γ (element of the K-basis) belongs to the τ_0-invariant subspace of $\mathscr{H}(\tau_0)$ generated by v. Indeed,

$$(v, \phi_\gamma)\phi_\gamma = \int_K \gamma(k^{-1})\tau(k)v\, dv$$

and

$$\tau(k)v = \tau_0(k)v .$$

(b) ⇒ (a): Let $0 \neq v \in \mathscr{H}(\tau_0)$. Let $\mathscr{H}_1$ be the τ_0-invariant subspace of $\mathscr{H}(\tau_0)$ generated by v. Then $\phi_\gamma \in \mathscr{H}_1$ for some $\gamma \in \mathscr{M}(\tau_0)$. Now, for some $g \in G$,

$$(\tau_0(g)\phi_\gamma, \phi_\delta) = \tau_{0,\delta,\gamma}(g) = \tau_{\delta,\gamma}(g) \neq 0 ,$$

so $\tau_0(g)\phi_\gamma$ and ϕ_δ are in $\mathscr{H}_1$. For each $\beta \in \mathscr{M}(\tau_0)$ we have $(\tau_0(g)\phi_\delta, \phi_\beta) = \tau_{\beta\delta}(g) \neq 0$ for some $g \in G$. Thus $\phi_\beta \in \mathscr{H}_1$ for all $\beta \in \mathscr{M}(\tau_0)$, so $\mathscr{H}_1 = \mathscr{H}(\tau_0)$.

(a) ⇒ (c): Suppose $\tau_{\gamma\delta} = 0$ for some $\gamma, \delta \in \mathscr{M}(\tau_0)$. Then, for all $g \in G$, $(\tau_0(g)\phi_\delta, \phi_\gamma) = 0$. Hence, the τ_0-invariant subspace of $\mathscr{H}(\tau_0)$ generated by ϕ_δ is orthogonal to ϕ_γ, so τ_0 is not irreducible.

(c) ⇒ (b): Clear. □

Let τ be K-multiplicity free, K being compact abelian. Define a relation $\prec$ on $\mathscr{M}(\tau)$ by: $\gamma \prec \delta$ iff $\tau_{\gamma,\delta} \neq 0$. Then $\gamma \prec \delta$ iff ϕ_γ is in the τ-invariant subspace of $\mathscr{H}(\tau)$ generated by ϕ_δ. It follows that

$$\beta \prec \gamma \text{ and } \gamma \prec \delta \Rightarrow \beta \prec \delta$$

Define a relation $\sim$ on $\mathscr{M}(\tau)$ by: $\gamma \sim \delta$ iff $\tau_{\gamma,\delta} \neq 0 \neq \tau_{\delta,\gamma}$. It follows that $\sim$ is an equivalence relation on $\mathscr{M}(\tau)$ and that, if $\tau_{\gamma,\delta} \neq 0, \alpha \sim \gamma, \beta \sim \delta$ then $\tau_{\alpha,\beta} \neq 0$. It follows that, for a given equivalence set, we can partition $\mathscr{M}(\tau)$ into three parts, the first part being the equivalence set, such that the corresponding 3×3 block matrix for $(\tau_{\gamma\delta}(g))$ takes the form (3.3). In view of Theorem 3.2 this proves:

THEOREM 3.3. *Let G be a lcsc. group with compact abelian subgroup K and let τ be a K-multiplicity free representation of G. Then there is a unique orthogonal decomposition of $\mathscr{H}(\tau)$ into subspaces $\mathscr{H}(\tau_i)$, where the τ_i's are precisely the irreducible subquotient representations of τ.*

3.2. THE CASE $SU(1, 1)$

For $\lambda \in \mathbf{C}$, $\xi = 0$ or $\frac{1}{2}$, the representation $\pi_{\xi,\lambda}$ of $G = SU(1, 1)$ on $L^2_\xi(K)$ (cf. (2.8)) is K-multiplicity free with K-content given by (2.13). By inspecting (2.29) for small but nonzero t and by using (2.24) it follows that

(3.4) $$\pi_{\xi,\lambda,m,n} \neq 0 \Leftrightarrow \pi_{\xi,\lambda,m,n}|_A \neq 0 \Leftrightarrow c_{\xi,\lambda,m,n} \neq 0\,;$$

where $c_{\xi,\lambda,m,n}$ is given by (2.30). Combination of (3.4) with Theorems 3.2 and 3.3 yields:

THEOREM 3.4. *Depending on* ξ *and* λ, *the representation* $\pi_{\xi,\lambda}$ *of* $SU(1,1)$ *has the following irreducible subquotient representations:*

(a) $\lambda + \xi \notin \mathbf{Z} + \frac{1}{2}$:

$\pi_{\xi,\lambda}$ *is irreducible itself.*

(b) $\lambda = 0, \xi = \frac{1}{2}$:

$\pi^{+}_{1/2,0}$ *on* $\mathrm{Cl\ Span}\ \{\phi_{1/2}, \phi_{3/2}, \ldots\}$,

$\pi^{-}_{1/2,0}$ *on* $\mathrm{Cl\ Span}\ \{\ldots, \phi_{-3/2}, \phi_{-1/2}\}$.

These are also subrepresentations.

(c) $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda > 0$:

$\pi^{+}_{\xi,\lambda}$ *on* $\mathrm{Cl\ Span}\ \{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \ldots\}$,

$\pi^{-}_{\xi,\lambda}$ *on* $\mathrm{Cl\ Span}\ \{\ldots, \phi_{-\lambda-3/2}, \phi_{-\lambda-1/2}\}$,

$\pi^{0}_{\xi,\lambda}$ *on* $\mathrm{Span}\ \{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \ldots, \phi_{\lambda-1/2}\}$.

Among these $\pi^{+}_{\xi,\lambda}$ *and* $\pi^{-}_{\xi,\lambda}$ *are subrepresentations.*

(d) $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda < 0$:

$\pi^{+}_{\xi,\lambda}$ *on* $\mathrm{Cl\ Span}\ \{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \ldots\}$,

$\pi^{-}_{\xi,\lambda}$ *on* $\mathrm{Cl\ Span}\ \{\ldots, \phi_{\lambda-3/2}, \phi_{\lambda-1/2}\}$,

$\pi_{\xi,\lambda}$ *on* $\mathrm{Span}\ \{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \ldots, \phi_{-\lambda-1/2}\}$.

Among these $\pi^{0}_{\xi,\lambda}$ *is a subrepresentation.*

Proof.

(a) $c_{\xi,\lambda,m,n} \neq 0$.

(b) $c_{1/2,0,m,n} \neq 0 \Leftrightarrow m, n \leqslant -\frac{1}{2}$ or $m, n \geqslant \frac{1}{2}$.

(c) $c_{\xi,\lambda,m,n} \neq 0 \Leftrightarrow -\lambda + \frac{1}{2} \leqslant n \leqslant \lambda - \frac{1}{2}$ or $m, n \leqslant -\lambda - \frac{1}{2}$ or $m, n \geqslant \lambda + \frac{1}{2}$.

Thus $c_{\xi, \lambda, m, n}$ has block matrix

$$\begin{array}{l} \\ m \leqslant -\lambda - \frac{1}{2} \\ -\lambda + \frac{1}{2} \leqslant m \leqslant \lambda - \frac{1}{2} \\ m \geqslant \lambda + \frac{1}{2} \end{array} \quad \begin{array}{ccc} n \leqslant -\lambda - \frac{1}{2} & -\lambda + \frac{1}{2} \leqslant n \leqslant \lambda - \frac{1}{2} & n \geqslant \lambda + \frac{1}{2} \\ \left(\begin{array}{c} * \\ 0 \\ 0 \end{array} \right. & \begin{array}{c} * \\ * \\ * \end{array} & \left. \begin{array}{c} 0 \\ 0 \\ * \end{array} \right) \end{array}$$

where each starred block has all entries nonzero.

(d) $c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \lambda + \frac{1}{2} \leqslant m \leqslant -\lambda - \frac{1}{2}$ or $m, n \leqslant \lambda - \frac{1}{2}$ or $m, n > -\lambda + \frac{1}{2}$. □

The finite-dimensional representation occurring in the above classification are the representations $\pi^0_{\xi, \lambda} (\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda \neq 0)$.

3.3. Notes

3.3.1. In the case of the unitary principal series (λ imaginary), Theorem 3.4 was first proved by Bargmann [2, sections 6 and 7]. See van Dijk [9, Theorem 4.1] for the statement and (infinitesimal) proof of our Theorem 3.4 in the general case. A proof of Theorem 3.4 similar to our proof was earlier given by Barut & Phillips [3, §II (4)].

3.3.2. Theorem 3.4 in the case of imaginary and nonzero λ is contained in a general theorem by Bruhat [5, Theorem 7; 2]: For $\xi \in \hat{M}$, $\lambda \in i\mathfrak{a}$, the principal series representation $\pi_{\xi, \lambda}$ of G (cf. (2.2)) is irreducible if $s \,.\, \lambda \neq \lambda$ for all $s \neq e$ in the Weyl group for (G, K).

3.3.3. Gelfand & Naimark [18, §5.4, Theorem 1] proved the irreducibility of the unitary principal series for $SL(2, \mathbf{C})$ by a global method different from ours, working in a noncompact realization and calculating the "matrix elements" of the representation with respect to a (continuous) $\bar{N}$-basis.

3.3.4. Analogues of Theorems 3.2 and 3.3 can be formulated in the case of non-abelian K, cf. [27, Theorem 3.3]. In that case the canonical matrix elements $\tau_{\gamma, \delta}$ are matrix-valued functions. By using this method, Naimark [34, Ch. 3, §9, No. 15] examined the irreducibility of the nonunitary principal series for $SL(2, \mathbf{C})$, see also Kosters [28].

3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of $\mathbf{R}^2$ and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of $SL(2, \mathbf{R})$ and [41, p. 560, Cor. 2] for the spherical principal series of $F_{4(-20)}$.

3.3.6. The method of this section does not show in an *a priori* way that a K-multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of K-multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let G be an lcsc. group.

Definition 4.1. Let σ and τ be Hilbert representations of G. The representation σ is called *Naimark related* to τ if there is a closed (possibly) unbounded) injective linear operator A from $\mathscr{H}(\sigma)$ to $\mathscr{H}(\tau)$ with domain $\mathscr{D}(A)$ dense in $\mathscr{H}(\sigma)$ and range $\mathscr{R}(A)$ dense in $\mathscr{H}(\tau)$ such that $\mathscr{D}(A)$ is σ-invariant and $A\sigma(g)v = \tau(G)Av$ for all $v \in \mathscr{D}(A)$, $g \in G$. Then we use the notation $\sigma \simeq \tau$ or $\sigma \overset{A}{\simeq} \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K-multiplicity free representations, K abelian.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathscr{H}(\sigma)$ onto $\mathscr{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathscr{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.