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and

$$\tau(k)v = \tau_0(k)v.$$

(b) \Rightarrow (a): Let $0 \neq v \in \mathcal{H}(\tau_0)$. Let \mathcal{H}_1 be the τ_0 -invariant subspace of $\mathcal{H}(\tau_0)$ generated by v . Then $\phi_\gamma \in \mathcal{H}_1$ for some $\gamma \in \mathcal{M}(\tau_0)$. Now, for some $g \in G$,

$$(\tau_0(g)\phi_\gamma, \phi_\delta) = \tau_{0, \delta, \gamma}(g) = \tau_{\delta, \gamma}(g) \neq 0,$$

so $\tau_0(g)\phi_\gamma$ and ϕ_δ are in \mathcal{H}_1 . For each $\beta \in \mathcal{M}(\tau_0)$ we have $(\tau_0(g)\phi_\delta, \phi_\beta) = \tau_{\beta\delta}(g) \neq 0$ for some $g \in G$. Thus $\phi_\beta \in \mathcal{H}_1$ for all $\beta \in \mathcal{M}(\tau_0)$, so $\mathcal{H}_1 = \mathcal{H}(\tau_0)$.

(a) \Rightarrow (c): Suppose $\tau_{\gamma\delta} = 0$ for some $\gamma, \delta \in \mathcal{M}(\tau_0)$. Then, for all $g \in G$, $(\tau_0(g)\phi_\delta, \phi_\gamma) = 0$. Hence, the τ_0 -invariant subspace of $\mathcal{H}(\tau_0)$ generated by ϕ_δ is orthogonal to ϕ_γ , so τ_0 is not irreducible.

(c) \Rightarrow (b): Clear. □

Let τ be K -multiplicity free, K being compact abelian. Define a relation $<$ on $\mathcal{M}(\tau)$ by: $\gamma < \delta$ iff $\tau_{\gamma, \delta} \neq 0$. Then $\gamma < \delta$ iff ϕ_γ is in the τ -invariant subspace of $\mathcal{H}(\tau)$ generated by ϕ_δ . It follows that

$$\beta < \gamma \text{ and } \gamma < \delta \Rightarrow \beta < \delta$$

Define a relation \sim on $\mathcal{M}(\tau)$ by: $\gamma \sim \delta$ iff $\tau_{\gamma, \delta} \neq 0 \neq \tau_{\delta, \gamma}$. It follows that \sim is an equivalence relation on $\mathcal{M}(\tau)$ and that, if $\tau_{\gamma, \delta} \neq 0, \alpha \sim \gamma, \beta \sim \delta$ then $\tau_{\alpha, \beta} \neq 0$. It follows that, for a given equivalence set, we can partition $\mathcal{M}(\tau)$ into three parts, the first part being the equivalence set, such that the corresponding 3×3 block matrix for $(\tau_{\gamma\delta}(g))$ takes the form (3.3). In view of Theorem 3.2 this proves:

THEOREM 3.3. *Let G be a lcsc. group with compact abelian subgroup K and let τ be a K -multiplicity free representation of G . Then there is a unique orthogonal decomposition of $\mathcal{H}(\tau)$ into subspaces $\mathcal{H}(\tau_i)$, where the τ_i 's are precisely the irreducible subquotient representations of τ .*

3.2. THE CASE $SU(1, 1)$

For $\lambda \in \mathbb{C}, \xi = 0$ or $\frac{1}{2}$, the representation $\pi_{\xi, \lambda}$ of $G = SU(1, 1)$ on $L^2_\xi(K)$ (cf. (2.8)) is K -multiplicity free with K -content given by (2.13). By inspecting (2.29) for small but nonzero t and by using (2.24) it follows that

$$(3.4) \quad \pi_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \pi_{\xi, \lambda, m, n}|_A \neq 0 \Leftrightarrow c_{\xi, \lambda, m, n} \neq 0,$$

where $c_{\xi, \lambda, m, n}$ is given by (2.30). Combination of (3.4) with Theorems 3.2 and 3.3 yields:

THEOREM 3.4. *Depending on ξ and λ , the representation $\pi_{\xi, \lambda}$ of $SU(1, 1)$ has the following irreducible subquotient representations:*

(a) $\lambda + \xi \notin \mathbf{Z} + \frac{1}{2}$:

$\pi_{\xi, \lambda}$ is irreducible itself.

(b) $\lambda = 0, \xi = \frac{1}{2}$:

$\pi_{1/2, 0}^+$ on $\text{Cl Span } \{\phi_{1/2}, \phi_{3/2}, \dots\}$,

$\pi_{1/2, 0}^-$ on $\text{Cl Span } \{\dots, \phi_{-3/2}, \phi_{-1/2}\}$.

These are also subrepresentations.

(c) $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda > 0$:

$\pi_{\xi, \lambda}^+$ on $\text{Cl Span } \{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \dots\}$,

$\pi_{\xi, \lambda}^-$ on $\text{Cl Span } \{\dots, \phi_{-\lambda-3/2}, \phi_{-\lambda-1/2}\}$,

$\pi_{\xi, \lambda}^0$ on $\text{Span } \{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \dots, \phi_{\lambda-1/2}\}$.

Among these $\pi_{\xi, \lambda}^+$ and $\pi_{\xi, \lambda}^-$ are subrepresentations.

(d) $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda < 0$:

$\pi_{\xi, \lambda}^+$ on $\text{Cl Span } \{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \dots\}$,

$\pi_{\xi, \lambda}^-$ on $\text{Cl Span } \{\dots, \phi_{\lambda-3/2}, \phi_{\lambda-1/2}\}$,

$\pi_{\xi, \lambda}$ on $\text{Span } \{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \dots, \phi_{-\lambda-1/2}\}$.

Among these $\pi_{\xi, \lambda}^0$ is a subrepresentation.

Proof.

(a) $c_{\xi, \lambda, m, n} \neq 0$.

(b) $c_{1/2, 0, m, n} \neq 0 \Leftrightarrow m, n \leq -\frac{1}{2}$ or $m, n \geq \frac{1}{2}$.

(c) $c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow -\lambda + \frac{1}{2} \leq n \leq \lambda - \frac{1}{2}$
or $m, n \leq -\lambda - \frac{1}{2}$ or $m, n \geq \lambda + \frac{1}{2}$.

Thus $c_{\xi, \lambda, m, n}$ has block matrix

$$\begin{array}{ccc}
 n \leq -\lambda - \frac{1}{2} & -\lambda + \frac{1}{2} \leq n \leq \lambda - \frac{1}{2} & n \geq \lambda + \frac{1}{2} \\
 \begin{array}{l} m \leq -\lambda - \frac{1}{2} \\ -\lambda + \frac{1}{2} \leq m \leq \lambda - \frac{1}{2} \\ m \geq \lambda + \frac{1}{2} \end{array} & \left(\begin{array}{ccc} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{array} \right) &
 \end{array}$$

where each starred block has all entries nonzero.

(d) $c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \lambda + \frac{1}{2} \leq m \leq -\lambda - \frac{1}{2}$ or $m, n \leq \lambda - \frac{1}{2}$
 or $m, n > -\lambda + \frac{1}{2}$. □

The finite-dimensional representation occurring in the above classification are the representations $\pi_{\xi, \lambda}^0(\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda \neq 0)$.

3.3. NOTES

3.3.1. In the case of the unitary principal series (λ imaginary), Theorem 3.4 was first proved by BARGMANN [2, sections 6 and 7]. See van DIJK [9, Theorem 4.1] for the statement and (infinitesimal) proof of our Theorem 3.4 in the general case. A proof of Theorem 3.4 similar to our proof was earlier given by BARUT & PHILLIPS [3, §II (4)].

3.3.2. Theorem 3.4 in the case of imaginary and nonzero λ is contained in a general theorem by BRUHAT [5, Theorem 7; 2]: For $\xi \in \hat{M}$, $\lambda \in ia$, the principal series representation $\pi_{\xi, \lambda}$ of G (cf. (2.2)) is irreducible if $s \cdot \lambda \neq \lambda$ for all $s \neq e$ in the Weyl group for (G, K) .

3.3.3. GELFAND & NAIMARK [18, §5.4, Theorem 1] proved the irreducibility of the unitary principal series for $SL(2, \mathbf{C})$ by a global method different from ours, working in a noncompact realization and calculating the “matrix elements” of the representation with respect to a (continuous) \bar{N} -basis.

3.3.4. Analogues of Theorems 3.2 and 3.3 can be formulated in the case of non-abelian K , cf. [27, Theorem 3.3]. In that case the canonical matrix elements $\tau_{\gamma, \delta}$ are matrix-valued functions. By using this method, NAIMARK [34, Ch. 3, §9, No. 15] examined the irreducibility of the nonunitary principal series for $SL(2, \mathbf{C})$, see also KOSTERS [28].