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3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of \mathbf{R}^2 and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of $SL(2, \mathbf{R})$ and [41, p. 560, Cor. 2] for the spherical principal series of $F_{4(-20)}$.

3.3.6. The method of this section does not show in an *a priori* way that a K -multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of K -multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let G be an lcsc. group.

Definition 4.1. Let σ and τ be Hilbert representations of G . The representation σ is called *Naimark related* to τ if there is a closed (possibly unbounded) injective linear operator A from $\mathcal{H}(\sigma)$ to $\mathcal{H}(\tau)$ with domain $\mathcal{D}(A)$ dense in $\mathcal{H}(\sigma)$ and range $\mathcal{R}(A)$ dense in $\mathcal{H}(\tau)$ such that $\mathcal{D}(A)$ is σ -invariant and $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathcal{D}(A)$, $g \in G$. Then we use the notation $\sigma \stackrel{A}{\simeq} \tau$ or $\sigma \simeq \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K -multiplicity free representations, K abelian.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathcal{H}(\sigma)$ onto $\mathcal{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathcal{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.

PROPOSITION 4.2. *Two unitary representations of an lcsc. group G are Naimark related if and only if they are unitarily equivalent.*

See WARNER [48, Prop. 4.3.1.4] for the proof.

Let K be a compact abelian subgroup of G . Let σ and τ be K -multiplicity free representations of G . Let $\{\phi_\delta\}$ and $\{\psi_\delta\}$ be K -bases for $\mathcal{H}(\sigma)$ and $\mathcal{H}(\tau)$, respectively.

LEMMA 4.3. *If $\sigma \stackrel{A}{\simeq} \tau$ then $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$, $\phi_\delta \in \mathcal{D}(A)$ and $\psi_\delta \in \mathcal{R}(A)$ ($\delta \in \mathcal{M}(\sigma)$), and there are nonzero complex numbers c_δ ($\delta \in \mathcal{M}(\sigma)$) such that*

$$(4.1) \quad (Av, \psi_\delta) = c_\delta (v, \phi_\delta), \quad v \in \mathcal{D}(A).$$

In particular

$$(4.2) \quad A\phi_\delta = c_\delta \psi_\delta.$$

Proof. Let $\delta \in \mathcal{M}(\sigma)$. Let $v \in \mathcal{D}(A)$. We have, by the intertwining property of A ,

$$\begin{aligned} \int_K \delta(k^{-1}) \sigma(k) v dk &= (v, \phi_\delta) \phi_\delta, \\ \int_K \delta(k^{-1}) A \sigma(k) v dk &= \int_K \delta(k^{-1}) \sigma(k) A v dk \\ &= \begin{cases} (Av, \psi_\delta) \psi_\delta & \text{if } \delta \in \mathcal{M}(\tau), \\ 0 & \text{if } \delta \notin \mathcal{M}(\tau). \end{cases} \end{aligned}$$

Since A is closed, we conclude that $(v, \phi_\delta) \phi_\delta \in \mathcal{D}(A)$ and

$$A((v, \phi_\delta) \phi_\delta) = \begin{cases} (Av, \psi_\delta) \psi_\delta & \text{if } \delta \in \mathcal{M}(\tau), \\ 0 & \text{if } \delta \notin \mathcal{M}(\tau). \end{cases}$$

Since A is injective with dense domain, the left hand side is nonzero for certain $v \in \mathcal{D}(A)$. Hence $\delta \in \mathcal{M}(\tau)$, $\phi_\delta \in \mathcal{D}(A)$ and (4.2) and (4.1) hold for certain nonzero c_δ . Finally, since A is closed with dense range, $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$. \square

LEMMA 4.4. *Let A be a possibly unbounded, not necessarily closed, injective linear operator from $\mathcal{H}(\sigma)$ to $\mathcal{H}(\tau)$ which satisfies all other properties of Definition 4.1. Suppose that $\phi_\delta \in \mathcal{D}(A)$ for all $\delta \in \mathcal{M}(\sigma)$, $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and,*

for each $\delta \in \mathcal{M}(\sigma)$, there is a complex number c_δ such that $(Av, \psi_\delta) = c_\delta(v, \phi_\delta)$ for all $v \in \mathcal{D}(A)$. Then the closure \bar{A} of A is one-valued and injective, \bar{A} satisfies all properties of Definition 4.1 and

$$(4.3) \quad \mathcal{D}(\bar{A}) = \left\{ v \in \mathcal{H}(\sigma) \mid \sum_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty \right\}.$$

Proof. Let $\{v_n\}$ be a sequence in $\mathcal{D}(A)$ such that $v_n \rightarrow v$ in $\mathcal{H}(\sigma)$ and $Av_n \rightarrow w$ in $\mathcal{H}(\tau)$. Then, for each $\delta \in \mathcal{M}(\sigma)$,

$$(w, \psi_\delta) = \lim_{n \rightarrow \infty} (Av_n, \psi_\delta) = c_\delta \lim_{n \rightarrow \infty} (v_n, \phi_\delta) = c_\delta(v, \phi_\delta).$$

Hence $v = 0$ iff $w = 0$, so \bar{A} is one-valued and injective.

To prove the domain invariance and intertwining property of \bar{A} , let

$$v \in \mathcal{D}(\bar{A}), \text{ so } v_n \rightarrow v, Av_n \rightarrow \bar{A}v$$

for some sequence $\{v_n\}$ in $\mathcal{D}(A)$. If $g \in G$ then

$$\sigma(g)v_n \rightarrow \sigma(g)v \text{ and } A\sigma(g)v_n = \tau(g)Av_n \rightarrow \tau(g)\bar{A}v,$$

so $\sigma(g)v \in \mathcal{D}(\bar{A})$ and $\bar{A}\sigma(g)v = \tau(g)\bar{A}v$.

Finally, to prove (4.3), first suppose that $v \in \mathcal{H}(\sigma)$ and

$$\sum_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty.$$

Then

$$\begin{aligned} v &= \sum (v, \phi_\delta)\phi_\delta, w := \sum c_\delta(v, \phi_\delta)\psi_\delta \in \mathcal{H}(\tau) \text{ and } \bar{A}\phi_\delta \\ &= c_\delta\psi_\delta, \text{ so, } w = \bar{A}v \text{ and } v \in \mathcal{D}(\bar{A}). \end{aligned}$$

Conversely, let $v \in \mathcal{D}(\bar{A})$. Then $\bar{A}v = \sum (\bar{A}v, \psi_\delta)\psi_\delta = \sum c_\delta(v, \phi_\delta)\psi_\delta$ (note $(\bar{A}v, \psi_\delta) = c_\delta(v, \phi_\delta)$ by (4.1)). Hence $\sum |c_\delta(v, \phi_\delta)|^2 < \infty$. \square

Next we will prove a criterium for Naimark relatedness of K -multiplicity free representations σ and τ in terms of the canonical matrix elements.

THEOREM 4.5. *Let G be an lcsc. group with compact abelian subgroup K . Let σ and τ be K -multiplicity free representations of G . Let $\{\phi_\delta\}$ and $\{\psi_\delta\}$ be K -bases of $\mathcal{H}(\sigma)$ and $\mathcal{H}(\tau)$, respectively. For each $\delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ let $0 \neq c_\delta \in \mathbb{C}$. Then the following two statements are equivalent:*

(a) $\sigma \stackrel{A}{\simeq} \tau$ and $A\phi_\delta = c_\delta\psi_\delta$, $\delta \in \mathcal{M}(\sigma)$.

(b) $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and, for all $\gamma, \delta \in \mathcal{M}(\sigma)$,

$$(4.4) \quad \tau_{\gamma, \delta} = C_{\gamma, \delta} \sigma_{\gamma, \delta}$$

with $C_{\gamma, \delta} = c_\gamma/c_\delta$.

If, moreover, σ and τ are irreducible then (a) and (b) are also equivalent to :

(c) For some $\gamma, \delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ (4.4) holds for some nonzero complex $C_{\gamma, \delta}$.

Proof.

(a) \Rightarrow (b): Apply Lemma 4.3. By using (4.1) we have

$$\begin{aligned} c_\gamma(\sigma(g)\phi_\delta, \phi_\gamma) &= (A\sigma(g)\phi_\delta, \psi_\gamma) = (\tau(g)A\phi_\delta, \psi_\gamma) \\ &= c_\delta(\tau(g)\psi_\delta, \psi_\gamma). \end{aligned}$$

(b) \Rightarrow (a): Define A on the domain $\{v \in \mathcal{H}(\sigma) \mid \sum |c_\delta(v, \phi_\delta)|^2 < \infty\}$ by $Av := \sum c_\delta(v, \phi_\delta)\psi_\delta$. Then A is injective with dense domain and range and A satisfies (4.1). We will prove that $\mathcal{D}(A)$ is G -invariant and that A is an intertwining operator. Let $v \in \mathcal{D}(A)$, $g \in G$. Then, by (4.4) and the definition of Av :

$$\begin{aligned} c_\gamma(\sigma(g)v, \phi_\gamma) &= c_\gamma \sum_\delta (v, \phi_\delta) \sigma_{\gamma, \delta}(g) \\ &= \sum_\delta c_\delta (v, \phi_\delta) \tau_{\gamma, \delta}(g) = (\tau(g)Av, \psi_\gamma). \end{aligned}$$

Hence

$$\sum_\gamma |c_\gamma(\sigma(g)v, \phi_\gamma)|^2 = \|\tau(g)Av\|^2 < \infty.$$

So $\sigma(g)v \in \mathcal{D}(A)$ and $A\sigma(g)v = \tau(g)Av$. Now apply Lemma 4.4.

(c) \Rightarrow (b): (σ, τ irreducible): We will first show that $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and, for each $\beta \in \mathcal{M}(\sigma)$, $\tau_{\gamma, \beta} = C_{\gamma, \beta} \sigma_{\gamma, \beta}$ and $\tau_{\beta, \delta} = C_{\beta, \delta} \sigma_{\beta, \delta}$ for some nonzero complex $C_{\gamma, \beta}$ and $C_{\beta, \delta}$. It follows from (4.4) evaluated for $g = g_1 k g_2$ that

$$\begin{aligned} &\sum_{\beta \in \mathcal{M}(\tau)} \beta(k) \tau_{\gamma, \beta}(g_1) \tau_{\beta, \delta}(g_2) \\ &= C_{\delta, \gamma} \sum_{\beta \in \mathcal{M}(\sigma)} \beta(k) \sigma_{\gamma, \beta}(g_1) \sigma_{\beta, \delta}(g_2), \quad g_1, g_2 \in G, k \in K. \end{aligned}$$

Both sides are absolutely and uniformly convergent Fourier series in $k \in K$. Because of Theorem 3.2 and the irreducibility of σ and τ , for each $\beta \in \mathcal{M}(\tau)$

respectively $\beta \in \mathcal{M}(\sigma)$ the Fourier coefficient at the left respectively right hand side does not vanish identically in g_1, g_2 . Hence $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and

$$\tau_{\gamma, \beta}(g_1)\tau_{\beta, \delta}(g_2) = C_{\gamma, \delta}\sigma_{\gamma, \beta}(g_1)\sigma_{\beta, \delta}(g_2).$$

This implies

$$\tau_{\gamma, \beta} = C_{\gamma, \beta}\sigma_{\gamma, \beta} \text{ and } \tau_{\beta, \delta} = C_{\beta, \delta}\sigma_{\beta, \delta} \text{ with } C_{\gamma, \beta}C_{\beta, \delta} = C_{\gamma, \delta}.$$

By repeating this argument we prove that $\tau_{\alpha, \beta} = C_{\alpha, \beta}\sigma_{\alpha, \beta}$ for all $\alpha, \beta \in \mathcal{M}(\sigma)$ and that $C_{\alpha, \beta}C_{\beta, \delta} = C_{\alpha, \delta}$, i.e. $C_{\alpha, \beta} = C_{\alpha, \delta}/C_{\beta, \delta}$. \square

COROLLARY 4.6. *Let G be an lcsc. group with compact abelian subgroup K . Then Naimark relatedness is an equivalence relation in the class of K -multiplicity free representations of G .*

4.2. THE CASE $SU(1, 1)$

Consider irreducible subquotient representations of $\pi_{\xi, \lambda}$ as classified in Theorem 3.4. By comparing K -contents it follows that the only possible nontrivial Naimark equivalences are:

$$\pi_{\xi, \lambda} \simeq \pi_{\xi, \mu}(\lambda + \xi, \mu + \xi \notin \mathbf{Z} + \frac{1}{2}, \lambda \neq \mu)$$

and

$$\begin{aligned} \pi_{\xi, \lambda}^+ &\simeq \pi_{\xi, -\lambda}^+, & \pi_{\xi, \lambda}^0 &\simeq \pi_{\xi, -\lambda}^0, & \pi_{\xi, \lambda}^- &\simeq \pi_{\xi, -\lambda}^- \\ & & & & & (\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda \neq 0). \end{aligned}$$

Suppose that σ and τ are irreducible subquotient representations of $\pi_{\xi, \lambda}$ and $\pi_{\xi, \mu}$, respectively, and that $\phi_m \in \mathcal{H}(\sigma) \cap \mathcal{H}(\tau)$ for some $m \in \mathbf{Z} + \xi$. It follows from Theorem 4.5 that $\sigma \simeq \tau$ iff $\tau_{\xi, \lambda, m, m} = \pi_{\xi, \mu, m, m}$. This last identity already holds if it is valid for the restrictions to A . In view of (2.29) and (2.30) we have: $\sigma \simeq \tau$ iff

$$(4.5) \quad \phi_{2i\lambda}^{(0, 2m)}(t) = \phi_{2i\mu}^{(0, 2m)}(t), \quad t \in \mathbf{R}.$$

Formula (4.5) holds if $\lambda = \pm\mu$ (cf. (2.26)). Conversely, assume (4.5) and expand both sides of (4.5) as a power series in $-(sh t)^2$ by using (2.23) and (2.20). The coefficients of $-(sh t)^2$ yield the equality

$$(m+1+\lambda)(m+1-\lambda) = (m+1+\mu)(m+1-\mu)$$

Hence $\lambda = \pm\mu$. We have proved: