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Proposition 2 may be expressed by a \prod_2^0 formula. First it is clear that we can construct a \sum_1^0 -formula ϕ_i that expresses the properties that

1. $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$ is a primitive recursive partition
2. $z_1 < z_2 < \dots < z_{n_k}$
3. $\{z_1, \dots, z_{n_k}\}$ is homogeneous for P_i
4. $k \leq n_k$
5. $2^{2^{z_1}} \leq n_k$

Proposition 2 asserts that for every k

$$\mathbb{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i .$$

V. CONSTRUCTION OF THE MODEL

We now have all the ingredients at hand to construct a non-standard model of Peano arithmetic, and we have only to assemble them according to the specifications of Section II.

Let P_i be an effective enumeration of all primitive recursive partitions $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$. By Proposition 2 we have that for every k

$$\mathbb{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i$$

where ϕ_i is the \sum_1^0 -formula of Section IV expressing the conditions (1)-(5) satisfied by the partition P_i .

Following the prescription given in Section III we let a_{kn_k} be the smallest number such that a_{k_1}, \dots, a_{kn_k} is an increasing sequence satisfying the formula $\bigwedge_{j \leq k} \phi_j$. Now we define the functions h_j by

$$h_0(k) = n_k \quad \text{for every } k$$

and for $j > 0$

$$h_j(k) = \begin{cases} a_{kj} & \text{for } j \leq n_k \\ h_{j-1}(k)^2 & \text{for } j > n_k . \end{cases}$$

Let $\mathcal{F} = \{f \mid f \leq h_j\}$.

Since $1 \leq h_0$ the function 1 is automatically in \mathcal{F} .

By Theorem 2 the sequence $\{h_j\}$ satisfies $\bigwedge_{j < \infty} \phi_j$ in \mathcal{F}/D . We now prove that this implies that the sequence $\{h_j\}$ satisfies the Stability and

Closure Conditions in \mathcal{F}/D . As we saw in Section III it suffices for this purpose to show that for each k

$$\begin{aligned} \mathbf{N} \vdash \exists z_1 \dots \exists z_{n_k} \bigwedge_{\substack{1 \leq i < j, j' < n_k \\ 1 \leq s < k}} [(\forall y < z_i) (\psi_s(y; z_j) \\ \leftrightarrow \psi_s(y; z_{j'}) \wedge z_{j-1}^2 < z_j)] \end{aligned} \quad (*)$$

Let t_i be the length of the sequence y in $\psi_i(y; z)$. Define the partitions $T: [\mathbf{N}]^2 \rightarrow 2$, $Q_i: \mathbf{N} \rightarrow t_i^2 + 1$, and $S_i: [\mathbf{N}]^{2e+1} \rightarrow 2$ by:

$$T(a, b) = \begin{cases} 1 & \text{if } a^2 < b \\ 0 & \text{if not} \end{cases}$$

$$Q_i(a) = \min(a, [t_i \log_2 a] + 1)$$

and for $a \in \mathbf{N}$, $c, c' \in [\mathbf{N}]^e$

$$S_i(a, c, c') = \begin{cases} 1 & \text{if } (\forall y < a) (\psi_i(y; c) \leftrightarrow \psi_i(y; c')) \\ 0 & \text{if not.} \end{cases}$$

The partitions T , Q_i , and S_i are clearly primitive recursive since $\psi_i(y; z)$ is a limited formula. Hence $T, Q_1, \dots, Q_k, S_1, \dots, S_k$ occur in the sequence $\{P_i\}$. Thus by looking sufficiently far in the sequence we can find a set $X_k = \{a_{k1}, \dots, a_{kn_k}\}$ which is homogeneous for $T, Q_1, \dots, Q_k, S_1, \dots, S_k$ with $n_k \geq k, 2^{2^{a_{k1}}}$.

Since $a_{kn_k} > a_{k1}^2$, $T(a_{k1}, a_{kn_k}) = 1$. Hence, by homogeneity,

$$T(a, b) = 1, \quad \text{i.e. } a^2 < b,$$

for all $a < b$ in X_k .

Since $\#X > 1$, and X_k is homogeneous for Q_i , $a_{k1} \geq t_i \log_2 a_{k1}$.

The number of sequences of numbers $< a_{k1}$ of length t_i is $< a_{k1}^{t_i}$. The number of distinct sequences of truth values of length $a_{k1}^{t_i}$ is $< 2^{a_{k1}^{t_i}}$. Now $n_k > 2^{2^{a_{k1}}} > 2^{a_{k1}^{t_i}}$ since $a_{k1} \geq t_i \log_2 a_{k1}$. Thus there are distinct $c, c' > a$ in X_k such that

$$(\forall y < a_{k1}) (\psi_i(y; c) \leftrightarrow \psi_i(y; c')),$$

i.e. $S_i(a_{k1}, c, c') = 1$.

By homogeneity

$$S_i(a, b, b') = 1 \quad \text{for all } a < b, b' \text{ in } X_k$$

proving (*).

We have thereby shown that \mathcal{F}/D is a model of the Peano axioms. Since a_{kn_k} was chosen minimal, Proposition 2 is false in \mathcal{F}/D , and hence independent of the Peano axioms.

Proposition 1 is also false in \mathcal{F}/D . In fact it is provable in Peano arithmetic that Proposition 1 implies Proposition 2. This is a consequence of the following lemma, provable in Peano arithmetic (c.f. Lemma 2.9 in [3]).

LEMMA 2. *Let $P_i : [\mathbb{N}]^e \rightarrow r_i, 1 \leq i \leq n$, be n partitions. There is a partition $P : [\mathbb{N}]^e \rightarrow r$ such that for all subsets H of \mathbb{N} of cardinality $> e$, H is homogeneous for P if and only if H is homogeneous for all the P_i .*

We may also obtain a purely finitary combinatorial principle which is false in our model.

PROPOSITION 3. *For all natural numbers e, r , and k there exists an N , such that for all partitions $P : [N]^e \rightarrow r$ there exists a subset X of N , with $\# X \geq k$ and $\# X \geq 2^{2^{\min X}}$, which is homogeneous for P .*

This result follows immediately from the infinite Ramsey Theorem by an application of König's Lemma. If we drop the condition that $\# X \geq 2^{2^{\min X}}$, then we obtain the usual finite Ramsey Theorem. Ramsey [11] gave a proof of the latter theorem which is formalizable in Peano arithmetic. Proposition 3 directly yields Proposition 1, for if $P : [\mathbb{N}]^e \rightarrow r$ is a partition and k is a number then by considering the partition $P \upharpoonright [N]^e$, where N is the number provided by Proposition 3 we obtain the required homogeneous set X for $P \upharpoonright [N]^e$ and hence for P . This proof may be carried out in Peano arithmetic. Thus, Proposition 3 is false in our model and independent of the Peano axioms.

VI. A SIMPLER MODEL

The condition in Proposition 1 that $\# X \geq 2^{2^{\min X}}$ can be simplified and so yield a simpler sequence $\{h_i\}$ of functions which define the model \mathcal{F}/D . In this section we describe such a model by using a combinatorial consequence of Ramsey's Theorem which is closer to the proposition proved independent in [3].

PROPOSITION 4. *Let $P : [\mathbb{N}]^e \rightarrow r$ be a primitive recursive partition. For every k there exists a finite subset X of \mathbb{N} , with $\# X \geq k$ and $\# X \geq \min X$, which is homogeneous for the partition P .*

Proposition 4 implies Proposition 1 via the following result, the proof of which is the same as the proof of Lemma 2.14 of [3].