

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 28 (1982)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NON-STANDARD MODELS OF PEANO ARITHMETIC
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Kapitel: VI. A Simpler Model
DOI: <https://doi.org/10.5169/seals-52238>

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We have thereby shown that \mathcal{F}/D is a model of the Peano axioms. Since a_{kn_k} was chosen minimal, Proposition 2 is false in \mathcal{F}/D , and hence independent of the Peano axioms.

Proposition 1 is also false in \mathcal{F}/D . In fact it is provable in Peano arithmetic that Proposition 1 implies Proposition 2. This is a consequence of the following lemma, provable in Peano arithmetic (c.f. Lemma 2.9 in [3]).

LEMMA 2. *Let $P_i : [\mathbb{N}]^e \rightarrow r_i$, $1 \leq i \leq n$, be n partitions. There is a partition $P : [\mathbb{N}]^e \rightarrow r$ such that for all subsets H of \mathbb{N} of cardinality $> e$, H is homogeneous for P if and only if H is homogeneous for all the P_i .*

We may also obtain a purely finitary combinatorial principle which is false in our model.

PROPOSITION 3. *For all natural numbers e , r , and k there exists an N , such that for all partitions $P : [N]^e \rightarrow r$ there exists a subset X of N , with $\# X \geq k$ and $\# X \geq 2^{2^{\min X}}$, which is homogeneous for P .*

This result follows immediately from the infinite Ramsey Theorem by an application of König's Lemma. If we drop the condition that $\# X \geq 2^{2^{\min X}}$, then we obtain the usual finite Ramsey Theorem. Ramsey [11] gave a proof of the latter theorem which is formalizable in Peano arithmetic. Proposition 3 directly yields Proposition 1, for if $P : [\mathbb{N}]^e \rightarrow r$ is a partition and k is a number then by considering the partition $P \upharpoonright [N]^e$, where N is the number provided by Proposition 3 we obtain the required homogeneous set X for $P \upharpoonright [N]^e$ and hence for P . This proof may be carried out in Peano arithmetic. Thus, Proposition 3 is false in our model and independent of the Peano axioms.

VI. A SIMPLER MODEL

The condition in Proposition 1 that $\# X \geq 2^{2^{\min X}}$ can be simplified and so yield a simpler sequence $\{h_i\}$ of functions which define the model \mathcal{F}/D . In this section we describe such a model by using a combinatorial consequence of Ramsey's Theorem which is closer to the proposition proved independent in [3].

PROPOSITION 4. *Let $P : [\mathbb{N}]^e \rightarrow r$ be a primitive recursive partition. For every k there exists a finite subset X of \mathbb{N} , with $\# X \geq k$ and $\# X \geq \min X$, which is homogeneous for the partition P .*

Proposition 4 implies Proposition 1 via the following result, the proof of which is the same as the proof of Lemma 2.14 of [3].

LEMMA 3. Let $P : [\mathbf{N}]^e \rightarrow r$ ($e \geq 2$) be a partition. There is a partition $P^* : [\mathbf{N}]^e \rightarrow r^*$ (where r^* depends only on $m, e,$ and r) such that if X^* is a finite subset of \mathbf{N} , homogeneous for P^* with $\# X^* \geq e+1$ and $\# X \geq \min X$, then the set $X = [\log_2 \log_2](X^*)$ is homogeneous for P , and

$$\# X \geq e + 1 \quad \text{and} \quad \# X \geq 2^{2^{\min X}}.$$

Moreover, if P is a primitive recursive partition, then P^* can be chosen to be primitive recursive.¹⁾

Since this proof that Proposition 4 implies Proposition 1 may be carried out in Peano arithmetic, it follows that Proposition 4 is also false in our model \mathcal{F}/D . However, our aim here is not merely to give a simple independent statement but to construct a simpler model for Peano arithmetic. Once again we actually use a version of the combinatorial principle which applies to several partitions. The following result is implied by Proposition 4 in Peano arithmetic.

PROPOSITION 5. Let $P_i : [\mathbf{N}]^{e_i} \rightarrow r_i$, $1 \leq i \leq n$, be a set of primitive recursive partitions. For every k there exists a finite subset of \mathbf{N} , with $\# X \geq k$ and $\# X \geq \min X$, which is simultaneously homogeneous for all the partitions P_1, \dots, P_n .

We now construct a non-standard model via Proposition 5. Let $\{P_i\}$ again be an effective enumeration of all the primitive recursive partitions $P_i : [\mathbf{N}]^{e_i} \rightarrow r_i$. Let c_{k1}, \dots, c_{kn_k} be an increasing sequence with c_{kn_k} the least number such that c_{k1}, \dots, c_{kn_k} is homogeneous for all P_1, \dots, P_k , with $c_{k1} \leq n_k$ and $k \leq n_k$. Define the functions g_j by

$$g_0(k) = n_k \quad \text{for every } k$$

and for $j > 0$

$$g_j(k) = \begin{cases} c_{kj} & \text{for } j \leq n_k \\ g_{j-1}(k)^2 & \text{for } j > n_k. \end{cases}$$

Let $\mathcal{F} = \{f \mid \exists j f \leq g_j\}$.

We shall show that \mathcal{F}/D is a model of Peano arithmetic by proving that there is an increasing sequence $\{h_j\}$ which lies in and is cofinal with \mathcal{F} and which satisfies the Stability and Closure Conditions. We set

$$h_j = [\log_2 \log_2 g_j].$$

¹⁾ Here, as is customary, $[x]$ is the greatest integer $\leq x$.

Since $h_j < g_j$, $h_j \in \mathcal{F}$. It follows from Lemma 2.13 of [3] that there is a primitive recursive partition R such that if X is homogeneous for R , with $\# X \geq \min X$ and $\# X \geq 3$, then for every $x, y, \in X$, $x < y$ implies $2^{2^x} < y$. Since this partition appears in the enumeration $\{P_i\}$ at some point k , it follows that, for all $i \geq k$ and $j < n_i$, $2^{2^{g_j(i)}} < g_{j+1}(i)$. Thus, if for a given j we choose an $m \geq k$ such that $n_m \geq j$, then, for all $i \geq m$, $2^{2^{g_j(i)}} < g_{j+1}(i)$. For every $i < m$ choose an s_i with $2^{2^{g_j(i)}} < g_{s_i}(i)$. Let

$$s = \max (s_1, \dots, s_{m-1}, j + 1)$$

Then

$$2^{2^{g_i}} < g_s .$$

Thus $h_s = [\log_2 \log_2 g_s] > g_j$, proving that $\{h_i\}$ is cofinal in \mathcal{F} .

For each partition P_k in the sequence $\{P_i\}$ there exists another partition $P_t (= P_k^*)$ satisfying the conditions of Lemma 3. By the definition of the functions g_j , the set $\{g_1(t), \dots, g_{n_t}(t)\}$ is homogeneous for P_t and $n_t \geq t$, $n_t \geq g_1(t)$. Hence, by Lemma 3, the set

$$\{h_1(t), \dots, h_{n_t}(t)\} = \{[\log_2 \log_2 g_1(t)], \dots, [\log_2 \log_2 g_{n_t}(t)]\}$$

is homogeneous for P_k and $n_t \geq 2^{2^{h_1(t)}}$. Thus, as in the previous section, the sequence $\{h_j\}$ fulfills the conditions which ensure the satisfaction of the Stability and Closure Conditions. This proves that \mathcal{F}/D is a model of the Peano axioms. Once again, since c_{kn_k} was chosen as minimal, it follows that Proposition 5, and hence Proposition 4, is false in \mathcal{F}/D , and therefore independent of Peano arithmetic.

As before we may formulate a finite version of this combinatorial principle.

PROPOSITION 6. *For every e, r , and k there exists an N such that for every partition $P : [N]^e \rightarrow r$ there exists a subset X of N , with $\# X \geq k$ and $\# X \geq \min X$, which is homogeneous for P .*

Again it is provable in Peano arithmetic that Proposition 6 implies Proposition 4, so that Proposition 6 is false in our model. Proposition 6 was first proved independent of Peano arithmetic in [3] by showing that it implies the consistency of Peano arithmetic and then applying Gödel's Theorem.

Let $C_k = \{i \mid i \leq c_{kn_k}\}$. The model \mathcal{F}/D is an initial segment not only of the ultrapower \mathbf{N}^I/D but also of the smaller ultraproduct $\prod_{k \in \mathbf{N}} C_k/D$.

This indicates that the function C given by $C(k) = c_{knk}$ is a very rapidly growing function. In fact the function C majorizes every recursive function which is a provably total function in Peano arithmetic.

THEOREM 5. *Let f be a recursive function. Let ψ be an elementary statement expressing the condition that f is a total function. If ψ is provable in Peano arithmetic, $f(k) < C(k)$ for all sufficiently large k .*

Proof. Suppose $t = \{k \mid f(k) \geq C(k)\}$ is infinite. Let D be a non-principal ultrafilter such that $t \in D$. Then $f^* \geq C^*$. On the other hand, $f^* = f(\mathbf{1}^*) \in \mathcal{F}/D$, so that $f^* < C^*$, a contradiction.

It follows a fortiori that if N is the smallest integer to satisfy Theorem 5 then this function N also majorizes every provably total recursive function (c.f. Theorem 3.2 in [3]).

We mentioned in the introduction that a by-product of our construction is a new proof of Specker's theorem that there exists a recursive partition with no recursively enumerable infinite homogeneous set. In fact we may obtain the stronger theorem that for each $e \geq 2$, there exists a primitive recursive partition: $P : [\mathbb{N}]^e \rightarrow 2$ such that P has no infinite homogeneous set in \sum_e^0 (c.f. Jockusch [10], Theorem 5.1). We outline the proof of this result. Let $\phi(y)$ be any formula. As in Section III, the limited associate $\hat{\phi}(y; z)$ of $\phi(y)$ defines a partition $P : [\mathbb{N}]^e \rightarrow 2$ such that every sequence $\{b_i\}$ of natural numbers homogeneous for P satisfies the Stability Condition for $\hat{\phi}(y; z)$ in \mathbb{N} . Hence, for any vector a in \mathbb{N} $\phi(a)$ holds in \mathbb{N} if and only if $\hat{\phi}(a; b)$ does. It follows that the set $\{a \mid \mathbb{N} \models \phi(a)\}$ is recursive in the set $\{b_i\}$. Thus the set $\{b_i\}$ is not in \sum_e^0 .

VII. VARIATIONS

We conclude with a series of remarks on various modifications of our construction.

(a) It is easily proved that if \mathcal{F} is closed under $<$ and contains $\mathbf{1}$, then \mathcal{F}/D is non-denumerable, for every non-principal ultrafilter D . Thus, this construction leads only to non-denumerable models. However, a slight variation of the basic construction yields denumerable models. Note that in the proof of Theorem 1 the function g is primitive recursive in f . It follows that we may define $\mathcal{F} = \{f \mid \exists j f \leq h_j \text{ and } f \text{ is primitive recursive}$