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This indicates that the function C given by $C(k) = c_{kn_k}$ is a very rapidly growing function. In fact the function C majorizes every recursive function which is a provably total function in Peano arithmetic.

THEOREM 5. *Let f be a recursive function. Let ψ be an elementary statement expressing the condition that f is a total function. If ψ is provable in Peano arithmetic, $f(k) < C(k)$ for all sufficiently large k.*

Proof. Suppose $t = \{ k \mid f(k) \geqslant C(k) \}$ is infinite. Let D be a non-principal ultrafilter such that $t \in D$. Then $f^* \geqslant C^*$. On the other hand, $f^* = f(\mathbf{1}^*) \in \mathscr{F}/D$, so that $f^* < C^*$, a contradiction.

It follows a fortiori that if N is the smallest integer to satisfy Theorem 5 then this function N also majorizes every provably total recursive function (c.f. Theorem 3.2 in [3]).

We mentioned in the introduction that a by-product of our construction is a new proof of Specker's theorem that there exists a recursive partition with no recursively enumerable infinite homogeneous set. In fact we may obtain the stronger theorem that for each $e \geqslant 2$, there exists a primitive recursive partition: $P : [\mathrm{N}]^e \rightarrow 2$ such that P has no infinite homogeneous set in $\sum_e^0$ (c.f. Jockusch [10], Theorem 5.1). We outline the proof of this result. Let $\phi(y)$ be any formula. As in Section III, the limited associate $\hat{\phi}(y; z)$ of $\phi(y)$ defines a partition $P : [\mathrm{N}]^e \rightarrow 2$ such that every sequence $\{ b_i \}$ of natural numbers homogeneous for P satisfies the Stability Condition for $\hat{\phi}(y; z)$ in N. Hence, for any vector a in N $\phi(a)$ holds in N if and only if $\hat{\phi}(a; b)$ does. It follows that the set $\{ a \mid \mathrm{N} \models \phi(a) \}$ is recursive in the set $\{ b_i \}$. Thus the set $\{ b_i \}$ is not in $\sum_e^0$.

VII. VARIATIONS

We conclude with a series of remarks on various modifications of our construction.

(a) It is easily proved that if $\mathscr{F}$ is closed under $<$ and contains $\mathbf{1}$, then $\mathscr{F}/D$ is non-denumerable, for every non-principal ultrafilter D. Thus, this construction leads only to non-denumerable models. However, a slight variation of the basic construction yields denumerable models. Note that in the proof of Theorem 1 the function g is primitive recursive in f. It follows that we may define $\mathscr{F} = \{ f \mid \exists j f \leqslant h_j$ and f is primitive recursive in $h_j\}$. The equivalence $f \equiv g$ in $\mathscr{F}$ defined by a non-principal ultrafilter D, viz $\{ i \mid f(i) = g(i) \} \in D$, may now be directly defined by a $\prod_2^0$-formula. This shows how to construct $\prod_2^0$-models of Peano arithmetic in the form of restricted ultrapowers.

(b) We may reduce the size of our models even further. Since in the proof of Theorem 1 the function g is defined from f by means of a limited formula g is even elementary recursive in f in the sense of Kalmar (see Kleene [11] §57 for a definition of elementary recursive functions). Thus, we may take $\mathscr{F}$ to consist of all functions which are elementary recursive in and majorized by a function h_j, for some j. Moreover, since the functions P_i from which the functions h_j are derived (Section IV) are defined by means of limited formulas, we may also take our sequence of partitions $\{ P_i \}$ to consist of elementary recursive partitions rather than primitive recursive partitions.

(c) It is possible to give the ultrapower a more algebraic appearance by switching from models of **N** to models of the ring **Z** of all integers. Let T be an axiom system for an ordered ring in which the non-negative elements obey the Peano axioms. Define the functions h_j as in Section V (or the g_j of Section VI). Let $\mathscr{F}$ be the ring of all functions $f : \mathbf{N} \to \mathbf{Z}$ such that, for some j, $|f| \leqslant h_j$. It is easily seen, as in Scott [6], that the ultrafilters are in one-one correspondence $D \leftrightarrow P_D$, with the minimal prime ideals P_D in $\mathscr{F}$, such that $\mathscr{F}/D = \mathscr{F}/P_D$. Principal ultrafilters correspond to principal prime ideals. Thus, we may construct non-standard models of **Z** by dividing the ring $\mathscr{F}$ by a non-principal minimal prime ideal P in $\mathscr{F}$. A non-standard model of **N** may then be selected as the set of those elements in $\mathscr{F}/P$ which are representable as the sum of four squares.

(d) It is possible to by-pass the Stability Condition in defining a non-standard model $\mathscr{F}/D$. It was condition (1) of Section III that assured us that $\mathscr{F}/D$ was a model of the axioms. We may define the family $\mathscr{F}$ to guarantee that condition (1) holds in a direct manner. We outline this approach now. Reduce the conjunction of the first k axioms to prenex normal form ϕ_k. We may associate with ϕ_k a sequence $f_{k1}, \ldots, f_{kn_k}$ of Skolem functions in the usual manner. For each k define the sequence of natural numbers $a_{k1}, \ldots, a_{kk}$ by induction. Let

$$a_{k1} = k + 1$$

and, for $1 < j < k$, let

$a_{k,j+1}$ = any number greater than a_{kj}^2 and the values of $f_{k1}, \ldots, f_{kn_k}$ as the arguments range over values $< a_{kj}$.

We now define the functions h_j as before by

$$h_j(k) = \begin{cases} a_{kj} & \text{for } j \leqslant k \\ h_{j-1}^2(k) & \text{for } j > k\,. \end{cases}$$

A simple induction argument now shows that (1) holds for the axioms and hence that $\mathscr{F}/D$ is a model of Peano arithmetic. If as in Theorem 3 the a_{kk} is chosen as the least number so that the sequence $a_{k1}, \ldots, a_{kk}$ satisfies the above conditions then a true statement which is false in $\mathscr{F}/D$ may be constructed via the method of Theorem 3.

The disadvantage of this direct approach is that the model $\mathscr{F}/D$ constructed in this manner is dependent in its definition upon logical formulas and so is not as purely an algebraic construction. Moreover the independent statement which results has no simple combinatorial expression as have those given in Sections IV, V, and VI. Note that in this approach we have not used the property peculiar to Peano's axioms concerning the limited associates of the axioms which is expressed in the proof of Theorem 4. This shows that the method outlined here applies to any recursively enumerable set of axioms for arithmetic which is sufficient to allow the coding required for Theorem 3. Thus, we may prove a general form of Gödel's Incompleteness Theorem without the use of self-reference techniques. At the same time the very generality of the approach outlined here indicates that there is no hope by this method to avoid the use of metamathematics. It is only the above-mentioned property of the Peano axioms vis-à-vis limited formulas that allowed us the latitude to define suitable functions h_j, and hence the model $\mathscr{F}/D$, by means of a combinatorial principal without reference to logical formulas.

Note (Added in proof)

The first sentence of the section entitled "Added in proof" of Kochen and Kripke [12] p. 294, which was inserted by the second author, is not correct and should be deleted in favor of the following corrected version. The first author proposed that the Paris-Harrington statement is false in an initial segment of any non-standard model, and this was verified jointly by the two authors. Adapting this idea, the second author defined the set $\mathscr{F}$ of functions which result in the model of Section V. The first author subsequently found the new set $\mathscr{F}$ of functions which define the simpler model of Section VI.