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ON BOOLEAN ALGEBRAS WITH DISTINGUISHED SUBALGEBRAS *

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In this paper, let $\mathscr{L} = \{+, \cdot, -, 0, 1, U\}$ be the language of Boolean algebras (BA's) with an additional unary predicate $\mathscr{U}$. Rubin has proved in [6] that the theory in $\mathscr{L}$ of Boolean algebras with a distinguished subalgebra (given by the interpretation of U) is undecidable. The main result of this paper is the solution of a problem stated in [6]: let **K** be the class of $\mathscr{L}$-structures $\mathscr{M} = (B, +, \cdot, -, 0, 1, A)$ where $(B, ...)$ is a complete BA (cBA), A is a complete subalgebra and the inclusion map from A to B is complete; we show that Th (**K**), the set of first- order $\mathscr{L}$-sentences which are true in every structure in **K**, is decidable. We shall abbreviate BA's $(B, ...)$ by their underlying set B.

The first idea to do this is to describe explicitly all completions of *Th* (**K**). One could then try to prove the decidability of *Th* (**K**) by Theorem 2 in [5]. A well-known example for a decidability proof in this style is given by the theory of BA's; the main task, to list all completions of this theory, was achieved by Tarski, see Theorem 5.5.10 in [1]. Before describing the complete first-order theory of a structure $\mathscr{M} = (B, A)$ in **K**, one has to get some idea how B "lies above A" and which details of the structure of an extension (B, A) of BA's can be expressed in first-order logic. Now B can be represented by the set of global sections of a sheaf of BA's over the Stone space X of A. Although the possibility of this representation is probably well-known to experts and although it is very easily established, it seems to give just the right intuition as to what are the relevant facts about the extension (B, A). We thus get an idea how to obtain a recursive set T of $\mathscr{L}$-sentences which looks rather natural and holds in every structure $\mathscr{M}$ of **K**.

It turns out that Comer's Feferman-Vaught-theorem on sheaves over Boolean spaces applies to the models of T. This establishes rather easily that a first-order sentence is in *Th* (**K**) if and only if it is provable from T

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and that Th (**K**) is decidable. It is then possible to describe the completions of T (which, however, was not necessary in the decidability proof).

As another example for the usefulness of sheaf representations of BA extensions (B, A), we consider the special case where B is finitely generated over A and we describe the action of a single automorphism of B leaving A pointwise fixed. This was motivated by Monk's paper [4] where the Galois group $\mathrm{Aut}_A B$ is studied in detail for a simple extension B of A and attempts are made towards finite extensions. The possibility of describing by a sheaf representation those extensions (S, R) of commutative rings for which the usual Galois correspondence can be established is, however, not new- see [8].

In section 1 of this paper, we give a sketch of the sheaf representation of a BA extension (B, A). We prove that the sheaf is Hausdorff iff A is relatively complete in B, which means that for $b \in B$, there is a largest $a \in A$ such that $a \leqslant b$.

In section 2, we provide a method to construct all automorphisms of B over A if B is a finite extension of A (2.4). We illustrate this method by computing the Galois group of B over A if A is relatively complete in B (2.6) and by proving in 2.7 that A is relatively complete in B iff there is a single automorphism of B over A moving every element of $B \setminus A$. This means that the finite extensions (B, A) where A is relatively complete in B are just the extensions called weakly Galois in [8].

Section 3 contains part of the machinery needed for the main result of the paper: if $(B, A) \in \mathbf{K}$, $\varphi (x_1 \dots x_n)$ is an $\mathscr{L}$-formula and $b_1, \dots, b_n \in B$, we prove that $\| \varphi [b_1 \dots b_n] \|$, the set of points p in the Stone space X of A such that φ is satisfied by $b_1 (p), \dots, b_n (p)$ in the stalk B_p over p, is a clopen subset of X. This enables us to apply the Feferman-Vaught theorem in Comer's version to our sheaf. More precisely, we show that there is an effective procedure assigning a formula $s_\varphi (yx_1 \dots x_n)$ to $\varphi (x_1 \dots x_n)$ such that the element a of A corresponding to $\| \varphi [b_1 \dots b_n] \|$ is the only element of A satisfying $s_\varphi (ab_1 \dots b_n)$ in (B, A). We then define the theory T in $\mathscr{L}$ and show that each $\mathscr{M}$ in **K** is a model of T.

Finally in section 4, we prove that the theorems of T are just the sentences in Th (**K**) and that Th (**K**) is decidable. We characterize elementary equivalence of T-models, give a list of all completions of T and prove that each of these completions has a model in **K**.

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1. The sheaf representation of Boolean algebra extensions

Let $\mathscr{L}$ be any language for first-order predicate logic. Suppose X is a non-empty set and for every $p \in X$ we have an $\mathscr{L}$-structure $\mathscr{B}_p = (B_p, \dots)$; put $S = \bigcup_{p \in X} B_p$. Suppose $\varphi(x_1 \dots x_n)$ is an $\mathscr{L}$-formula, $u \subseteq X$ and $f_1, \dots, f_n : u \to S$ are such that $f_i(p) \in B_p$ for $1 \leqslant i \leqslant n$ and $p \in u$. Then let

$$\| \varphi [f_1 \dots f_n] \| = \{ p \in u \mid \mathscr{B}_p \models \varphi [f_1(p) \dots f_n(p)] \} .$$

We may think of $\| \varphi [f_1 \dots f_n] \| \subseteq X$ as being a (Boolean) truth value of $\varphi [f_1 \dots f_n]$ in the power set of X.

A sheaf of $\mathscr{L}$-structures is a sequence

$$\mathscr{S} = (S, \pi, X, \mu)$$

such that a) S and X are topological spaces and $\pi : S \to X$ is a continuous open local homeomorphism from S onto X, b) μ is a function assigning to each $p \in X$ an $\mathscr{L}$-structure $\mathscr{B}_p = (B_p, \dots)$ such that the B_p are pairwise disjoint, $S = \bigcup_{p \in X} B_p$ and $\pi(s) = p$ iff $s \in B_p$, c) for every open subset u of X and continuous $f_1, \dots, f_n : u \to S$ satisfying $f_i(p) \in B_p$ for $p \in u$ and every atomic $\mathscr{L}$-formula $\varphi(x_1 \dots x_n)$, $\| \varphi [f_1 \dots f_n] \|$ is an open subset of u.

The $\mathscr{L}$-structure $\mathscr{B}_p$ is called the stalk of $\mathscr{S}$ over p. — Let, if $\mathscr{S}$ is a sheaf of $\mathscr{L}$-structures, $\Gamma(\mathscr{S})$ be the set of all continuous functions $f : X \to S$ satisfying $f(p) \in B_p$ for $p \in X$ (the set of "global sections" of $\mathscr{S}$). So $\Gamma(\mathscr{S})$ is, if non-empty, (the underlying set of) a substructure of the product structure $\prod_{p \in X} \mathscr{B}_p$, hence an $\mathscr{L}$-structure.

For the rest of the paper, let $\mathscr{L} = \{ +, \cdot, -, 0, 1, U \}$ where U is a unary predicate. We indicate how, for a given BA extension (B, A), B may be represented by $\Gamma(\mathscr{S})$ where $\mathscr{S}$ is a sheaf of $\mathscr{L}$-structures over a Boolean space. We omit most of the proofs which are easy and entirely analoguous to well-known representation theorems for lattices over Boolean spaces. Let X be the Stone space of A, i.e. the set of all ultrafilters of A with the usual topology. For $p \in X$, let $< p >$ be the filter of B generated by p. Let $\pi_p : B \to B / < p > = B_p$ be the canonical epimorphism. So B_p is a BA with at least two elements. For $p, q \in X$ and $p \neq q$, B_p and B_q are disjoint. Let $S = \bigcup_{p \in X} B_p$ and $\pi : S \to X$ be defined as stated in b) above. Let, for $p \in X$, $\mu(p)$ be the $\mathscr{L}$-structure $(B_p, \dots, \{0, 1\})$. For $u \subseteq X$ open and $b \in B$, let $M_{ub} = \{ \pi_p(b) \mid p \in u \}$. The set of these M_{ub} constitutes a base for a topology of S, and this makes $\mathcal{S} = (S, \pi, X, \mu)$ a sheaf of $\mathcal{L}$-structures. Furthermore, for $b \in B$, $\sigma_b : X \to S$ defined by $\sigma_b(p) = \pi_p(b)$ is a global section of $\mathcal{S}$ and

$$\left.\begin{aligned} \sigma : B &\to \Gamma(\mathcal{S}) \\ b &\mapsto \sigma_b \end{aligned}\right\}$$

is an isomorphism from B onto $\Gamma(\mathcal{S})$. We shall now identify B with $\Gamma(\mathcal{S})$; so every $b \in B$ is a function from X to S. This identifies A with those $b \in B$ such that for every $p \in X$ $b(p) = 0$ or $b(p) = 1$, i.e. with those $b \in B$ satisfying $\| U(b) \| = X$. Let C be the BA of clopen subsets of X and $e(c)$ the characteristic function of c for $c \in C$. Thus e is an isomorphism from C onto $A \subseteq B$.

In the rest of this section, we show that the property of being a Hausdorff sheaf for $\mathcal{S}$ is equivalent to a property of the extension (B, A) which reflects, in a way which is first-order expressible in $\mathcal{L}$, completeness of the embedding of A into B. Recall that, for a sheaf $\mathcal{S}$ over a Boolean space X, S is a T_2-space iff, for any $f, g \in \Gamma(\mathcal{S})$, $\| f = g \|$ is a clopen subset of X; $\mathcal{S}$ is then said to be a Hausdorff sheaf. Call A relatively complete in B if, for every $b \in B$, there is a largest element $a \in A$ such that $a \leqslant b$, equivalently: for $b \in B$, there is a largest $a \in A$ such that $a \cdot b = 0$ or: for $b \in B$, there is a smallest $a \in A$ such that $b \leqslant a$.

1.1. Proposition. *$\mathcal{S}$ is a Hausdorff sheaf iff A is relatively complete in B.*

Proof. Suppose $\mathcal{S}$ is Hausdorff and $b \in B$. Let $d \in B$ such that $d(p) = 0$ for every $p \in X$, let $c = \| b = d \|$ and $a = e(c)$. Then a is the largest element of A satisfying $a \cdot b = 0$.

Conversely, let A be relatively complete in B and suppose $f, g \in B$. Let a be the largest element of A such that $a \leqslant f \cdot g + -f \cdot -g$. Let $c \in C$ such that $a = e(c)$. Then $\| f = g \| = c$ is a clopen subset of X.

1.2. Remark. Let A be relatively complete in B. Then the inclusion map from A to B is a complete homomorphism.

Proof. Suppose M is a subset of A having a supremum a in A. We show that a is the supremum of M in B. Clearly, a is an upper bound for M in B. Suppose that b is another upper bound for M in B. Let $\alpha \in A$ be the largest element of A such that $\alpha \leqslant b$. For every $m \in M \subseteq A$, we have $m \leqslant b$, hence $m \leqslant \alpha$ and $a \leqslant \alpha \leqslant b$.

The following facts are trivial:

1.3. REMARK. *a)* Let A and the inclusion map from A to B be complete. Then A is relatively complete in B.

b) Suppose A is relatively complete in B and B is complete. Then A is complete.

2. RELATIVE AUTOMORPHISMS OF FINITE EXTENSIONS

We first give an internal description of a finite extension (B, A) where $B = A(u_1 \dots u_n)$ and $n \in \omega$. We shall always assume that $u_1, \dots, u_n$ are the atoms of the subalgebra of B generated by $u_1, \dots, u_n$; i.e. that they are non-zero, pairwise disjoint and $u_1 + \dots + u_n = 1$. Let $I_r = \{a \in A \mid a \cdot u_r = 0\}$ for $1 \leqslant r \leqslant n$. Clearly, each I_r is a proper ideal of A and $I_1 \cap \dots \cap I_n = \{0\}$. The family $(I_r \mid 1 \leqslant r \leqslant n)$ completely characterizes the extension (B, A):

2.1. REMARK. Suppose $C = A(v_1 \dots v_n)$ is a finite extension of A where $v_1, \dots, v_n$ are pairwise disjoint and $1 = v_1 + \dots + v_n$. Let $B = A(u_1 \dots u_n)$ be as above. There is an isomorphism g from B onto C satisfying $g(a) = a$ for $a \in A$ and $g(u_r) = v_r$ iff, for each r, $\{a \in A \mid a \cdot v_r = 0\} = I_r$.

Proof. By Theorem 12.4 in [7].

2.2. REMARK. A is relatively complete in $B = A(u_1 \dots u_n)$ iff, for each r, I_r is a principal ideal.

Proof. The only—if part follows by the definition of relative completeness. Now suppose $\alpha_r \in A$ generates I_r; let $b \in B$ and $I = \{a \in A \mid a \cdot b = 0\}$. There are $a_1, \dots, a_n \in A$ such that $b = a_1 \cdot u_1 + \dots + a_n \cdot u_n$. It follows that I is the principal ideal generated by $\alpha = (-a_1 + \alpha_1) \cdot \dots \cdot (-a_n + \alpha_n)$.

Conversely, given any family $(I_r \mid 1 \leqslant r \leqslant n)$ of proper ideals in A satisfying $I_1 \cap \dots \cap I_n = \{0\}$, there is an extension $A(u_1 \dots u_n)$ of A such that $I_r = \{a \in A \mid a \cdot u_r = 0\}$: let $D = A(x_1 \dots x_n)$ be the free product of A and a finite BA with atoms $x_1, \dots, x_n$. Let

$$K = \{i_1 \cdot x_1 + \dots + i_n \cdot x_n \mid i_1 \in I_1, \dots, i_n \in I_n\}.$$

K is an ideal of D; the canonical epimorphism π from D onto $B = D/K$ is one- one on A, and for $a \in A$, $\pi(a) \cdot u_r = 0$ iff $a \in I_r$ where $u_r = \pi(x_r)$. Now identify A with the subalgebra $\pi(A)$ of B.

For the rest of this section we think, as in section 1, of B as being the set of global sections of a sheaf $\mathscr{S} = (S, \pi, X, \mu)$ of Boolean algebras over a

Boolean space X; we use the abbreviations of section 1. For $p \in X$, $B_p = \{b(p) \mid b \in B\}$. Since $b(p) \in \{0, 1\}$ for $b \in A$ and $B = A(u_1 \ldots u_n)$, B_p is a finite BA with atoms $\{u_r(p) \mid 1 \leqslant r \leqslant n\} \setminus \{0\}$.

Let $G = \mathrm{Aut}_A B$ be the group of those automorphisms of B leaving A pointwise fixed, i.e. G is the Galois group of B over A. Suppose $g \in G$ and $p \in X$. Since $g(a) = a$ for $a \in A$, g induces an automorphism of B_p which, in turn, is induced by a permutation of the (at most n) atoms of B_p. This gives rise to the following definitions (S_n is the group of permutations of $\{1, \ldots, n\}$).

Let $p \in X$. For $1 \leqslant r, l \leqslant n$, say $u_r \sim u_l$ at p if there is a neighbourhood u of p such that, for $q \in u$, $u_r(q) = 0$ iff $u_l(q) = 0$. $\pi \in S_n$ is said to be compatible with p if $u_r \sim u_{\pi(r)}$ at p for $1 \leqslant r \leqslant n$. $g \in G$ is said to be induced by π at p if $g(u_r)(p) = u_{\pi(r)}(p)$ for $1 \leqslant r \leqslant n$. Note that, if one of these definitions holds (for fixed $u_r, u_l, \pi \in S_n, g \in G$) for some $p \in X$, then it holds (for the same $u_r, u_l, \pi \in S_n, g \in G$) for every q in some neighbourhood of p. And $u_r \sim u_l$ at p means that there is a clopen subset c of X such that $p \in c$ and, for $a \in A$ satisfying $a \leqslant e(c)$, $a \in I_r$ iff $a \in I_l$.

2.3. Lemma. Suppose $p \in X$ and $\pi \in S_n$. Then π is compatible with p iff there is some $g \in G$ which is induced by π at p.

Proof. Suppose π induces g at p and $1 \leqslant r \leqslant n$. Let u be a neighbourhood of p such that $g(u_r)(q) = u_{\pi(r)}(q)$ for $q \in u$. Thus, for $q \in u$, $u_{\pi(r)}(q) = 0$ iff $g(u_r)(q) = 0$ iff $u_r(q) = 0$ since g induces an automorphism of B_q.

Conversely, suppose π is compatible with p. Choose a clopen neighbourhood c of p such that $u_r(q) = 0$ iff $u_{\pi(r)}(q) = 0$ for $1 \leqslant r \leqslant n$ and $q \in u$. Let $a = e(c)$. By 2.1 and the remark preceding this lemma, there is some $g \in G$ such that $g(u_r) = -a \cdot u_r + a \cdot u_{\pi(r)}$ for every r. This g is induced by π at p, since $a(p) = 1$ and hence $g(u_r)(p) = u_{\pi(r)}(p)$.

2.4. Theorem. a) *Let* $X = \cup \{c_\pi \mid \pi \in S_n\}$ *be a partition of* X *into pairwise disjoint clopen subsets such that, for every* $p \in c_\pi$, π *is compatible with* p. *Put* $a_\pi = e(c_\pi)$ *for* $\pi \in S_n$. *Then there is* $g \in G$ *such that, for* $1 \leqslant r \leqslant n$,

$$g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}.$$

b) *Conversely, let* $g \in G$. *Then there is a partition* $X = \cup \{c_\pi \mid \pi \in S_n\}$ *of* X *into pairwise disjoint clopen subsets such that, for* $p \in c_\pi$, π *is compatible with* p, *and* $g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}$, *where* $a_\pi = e(c_\pi)$.

Proof. First note that $g \in G$, $a_\pi = e(c_\pi)$ where $(c_\pi \mid \pi \in S_n)$ is a partition of X and $g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}$ imply that π is compatible with p for $p \in c_\pi$: by $p \in c_\pi$, we get $a_\pi(p) = 1$ and $a_\rho(p) = 0$ for $\rho \in S_n$, $\rho \neq \pi$. So $g(u_r)(p) = u_{\pi(r)}(p)$, g is induced by π at p, and π is compatible with p.

To prove a), note that $\{a_\pi \cdot u_r \mid \pi \in S_n, 1 \leqslant r \leqslant n\}$ is a set of pairwise disjoint elements of B with supremum 1 and generating B over A. The existence of g follows by 2.1 and the remark preceding 2.3.

To prove b), let $g \in G$. For $\pi \in S_n$, put

$$v_\pi = \{p \in X \mid \pi \text{ induces } g \text{ at } p\} .$$

Each v_π is an open subset of X, and $X = \cup \{v_\pi \mid \pi \in S_n\}$: suppose $p \in X$. Define $\pi \in S_n$ as follows: let $1 \leqslant r \leqslant n$. If $u_r(p) = 0$, then $g(u_r)(p) = 0$; put $\pi(r) = r$. If $u_r(p) \neq 0$, $u_r(p)$ and hence $g(u_r)(p)$ is an atom of B_p; let $\pi(r) = l$ where $g(u_r)(p) = u_l(p)$. Clearly, $p \in v_\pi$.

Since X is a Boolean space, there is a family $(c_\pi \mid \pi \in S_n)$ such that c_π is a clopen subset of v_π, $X = \cup \{c_\pi \mid \pi \in S_n\}$ and the c_π are pairwise disjoint. Put $a_\pi = e(c_\pi)$. Suppose $1 \leqslant r \leqslant n$ and $p \in X$, e.g. $p \in c_\pi$. Then $p \in v_\pi$ and

$$(\sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\})(p) = g(u_r)(p) .$$

Theorem 2.4 says that the automorphisms of B over A are completely determined by certain partitions $(a_\pi \mid \pi \in S_n)$ of A resp. $(c_\pi \mid \pi \in S_n)$ of C. Unfortunately, for a given $g \in G$, a partition $(c_\pi \mid \pi \in S_n)$ defining g is not uniquely determined, since there may be different possibilities of choosing a clopen disjoint refinement of $(v_\pi \mid \pi \in S_n)$. We conclude this section by illustrating 2.4 by several examples.

If H is any group and A a BA, let X be the Stone space of A and

$$H[A] = \{f : X \to H \mid f \text{ is continuous}\}$$

where H is given the discrete topology. $H[A]$ is a subgroup of H^X and is usually called the bounded Boolean power of H by A. Recall that, for $B = A(u_1 \dots u_n)$, A and the subalgebra of B generated by $u_1, \dots, u_n$ are independent iff $a \cdot u_r \neq 0$ for $a \in A \setminus \{0\}$, $1 \leqslant r \leqslant n$. A is then relatively complete in B. Conversely, suppose A is relatively complete in B. Then there is a partition $(a_k \mid 1 \leqslant k \leqslant n)$ of A (some of the a_k may equal zero) such that, for each k, the relative algebra $B \restriction a_k = \{x \in B \mid x \leqslant a_k\}$ is generated over $A \restriction a_k$ by k disjoint elements $v_1, \dots, v_k$ which are independent from $A \restriction a_k$: for $1 \leqslant r, l \leqslant n$, the set of those $p \in X$ such that $u_r(p) = u_l(p)$ is clopen. Hence, for $1 \leqslant k \leqslant n$, $c_k = \{p \in X \mid B_p \text{ has exactly } k \text{ atoms}\}$ is clopen; put $a_k = e(c_k)$. By a compactness argument, construct $v_1, \ldots, v_k \in B \restriction a_k$ by patching together some of the u_r such that for $p \in c_k$, the atoms of B_p are $v_1(p), \ldots, v_k(p)$.

2.5. EXAMPLE. Suppose $a \cdot u_r \neq 0$ for $1 \leqslant r \leqslant n$ and $a \in A \setminus \{0\}$. Then $\mathrm{Aut}_A B \cong S_n[A]$.

Proof. Our assumption says that $u_r(p) \neq 0$ for each r and each $p \in X$. Hence each $\pi \in S_n$ is compatible with each $p \in X$ and, for fixed $g \in G$, the open sets v_π in the proof of 2.4 are disjoint, hence $c_\pi = v_\pi$. An isomorphism $\varphi : G \to S_n[A]$ is established by defining $\varphi(g)(p) = \pi$ iff $p \in v_\pi$.

2.6. EXAMPLE. Suppose A is relatively complete in B. Then there is a partition $(a_k \mid 1 \leqslant k \leqslant n)$ of A such that

$$\mathrm{Aut}_A B \cong S_1[A \restriction a_1] \times \ldots \times S_n[A \restriction a_n].$$

Proof. Choose, for $1 \leqslant k \leqslant n$, $a_k \in A$ as indicated above and let G_k be the Galois group of $B \restriction a_k$ over $A \restriction a_k$. Clearly,

$$\mathrm{Aut}_A B \cong G_1 \times \ldots \times G_n,$$

since $a_k \in A$. By 2.5, $G_k \cong S_k[A \restriction a_k]$.

2.7. PROPOSITION. *The following conditions on* (B, A) *are equivalent:*

a) *A is relatively complete in B;*

b) *there is some $g \in G$ such that $g(b) \neq b$ for $b \in B \setminus A$;*

c) *there is some finite subgroup H of G such that, for every $b \in B \setminus A$, there is some $g \in H$ satisfying $g(b) \neq b$.*

Proof. Assume a). There is a finite partition T of C such that, for $1 \leqslant r \leqslant n$, $t \in T$ and $p, q \in t$, $u_r(p) = 0$ iff $u_r(q) = 0$. For $t \in T$, let $\pi_t \in S_n$ such that, for $p \in t$, $\pi_t(r) = r$ if $u_r(p) = 0$ and $u_r(p) \mapsto u_{\pi_t(r)}(p)$ is a cyclic permutation of the atoms of B_p which moves all these atoms. π_t is compatible with each $p \in t$; hence there is some $g \in G$ such that g is induced by π_t for $p \in t$, $t \in T$. Now let $b \in B \setminus A$. Choose $p \in X$, e.g. $p \in t$ where $t \in T$, such that $b(p) \notin \{0, 1\}$; put $b' = g(b)$. Let $At(B_p)$ be the set of atoms of B_p, $M = \{\alpha \in At(B_p) \mid \alpha \leqslant b(p)\}$, g_p the automorphism of B_p induced by g, $M' = \{g_p(\alpha) \mid \alpha \in M\}$. By the choice of π_t and g,

$$b'(p) = g_p(b(p)) = \sum M' \neq \sum M = b(p)$$

which proves $b' \neq b$ – since, if π is a cyclic permutation of a finite set Y moving every element of Y and $M \subseteq Y$ satisfies $M = \{\pi(m) \mid m \in M\}$, then $M = \phi$ or $M = Y$.

To prove that b) implies c) it is sufficient to know that every finitely generated subgroup of G is finite. We indicate a construction for finite subgroups of G. Let $T \subseteq C$ be a finite partition of C. A function $\varphi : T \to S_n$ is said to be compatible if, for every $t \in T$ and $p \in t$, $\varphi(t)$ is compatible with p. For each compatible $\varphi : T \to S_n$ let g_φ be the element of G mapping u_r to $\sum \{e(t) \cdot u_{\varphi(t)(r)} \mid t \in T\}$. It is easily seen that

$$G_T = \{g_\varphi \mid \varphi : T \to S_n \text{ compatible}\}$$

is a finite subgroup of G and that every finite subset of G is contained in some G_T.

Now suppose c), i.e. there is some finite subgroup H of G moving every $b \in B \setminus A$. We may assume that $H = G_T$ for some finite partition T of C. Assume that A is not relatively complete in B. By 2.2 there is some r such that I_r is not a principal ideal; w.l.o.g., $r = 1$. Let $\sigma = \{p \in X \mid u_1(p) = 0\}$. σ is a subset of X which is open but not closed; choose $p \in X$ which lies in the closure of σ but not in σ. W.l.o.g., for some k satisfying $1 \leqslant k > n$,

$$\{r \mid 1 \leqslant r \leqslant n \text{ and } u_r \sim u_1 \text{ at } p\} = \{1, ..., k\}.$$

Let c be a clopen neighbourhood of p such that, for $1 \leqslant r \leqslant k$ and $q \in c$, $u_r(q) = 0$ iff $u_1(q) = 0$. W.l.o.g., $c \in T$. There is some l such that $k < l \leqslant n$ and $u_l(p) \neq 0$; otherwise, let $c' \subseteq c$ a neighbourhood of p such that $u_l(q) = 0$ for $q \in c'$ and $k < l \leqslant n$. Choose $q \in c' \cap \sigma$ (since p lies in the closure of σ). In B_q, which has at least two elements, $1 = u_1(q) + ... + u_n(q) = 0 + ... + 0 = 0$, a contradiction. — Put $a = e(c)$ and $b = a \cdot u_1 + ... + a \cdot u_k$. $b \in B \setminus A$, since $0 < b(p) = u_1(p) + ... + u_k(p) < 1$ by our preceding claim. We prove that, for $g \in H = G_T$, $g(b) = b$, thus arriving at a final contradiction: there is some compatible $\varphi : T \to S_n$ such that $g = g_\varphi$. Consider $k \leqslant n$, $c \in T$ and $p \in c$ as constructed above. Since φ is compatible, $\pi = \varphi(c)$ is compatible with p; hence π maps the set $\{1, ..., k\}$ into itself, $g_\varphi(a \cdot u_r) = a \cdot u_{\pi(r)}$ for $1 \leqslant r \leqslant k$ (where $a = e(c)$) and $g(b) = b$.

3. Truth values in A for statements about (B, A)

For the rest of this paper, let $\mathscr{L}_{BA} = \{+, \cdot, -, 0, 1\}$ the language of BAs and $\mathscr{L} = \mathscr{L}_{BA} \cup \{U\}$. Let T_{BAU} be the theory in $\mathscr{L}$ such that the models of T_{BAU} have the form $(B, +, \cdot, -, 0, 1, A)$ where $(B, \ldots)$ is a BA and A is a subalgebra of B. We abbreviate a model $(B, \ldots, A)$ of T_{BAU} by $\mathscr{M} = (B, A)$. We assume the construction and notations of section 1. For each $\mathscr{L}$-formula $\varphi(x_1 \ldots x_n)$ and $b_1, \ldots, b_n \in B$, we defined

$$\| \varphi [b_1 \ldots b_n] \| = \{ p \in X \mid B_p \models \varphi [b_1(p) \ldots b_n(p)] \}$$

where B_p abbreviates $(B_p, 2)$ and 2 is the two-element BA. Our first claim is that if $c = \| \varphi [b_1 \ldots b_n] \|$ is a clopen subset of X for every φ, then $e(c) \in A$ is first-order definable in $\mathscr{M} = (B, A)$ from the parameters $b_1, \ldots, b_n \in B$:

3.1. Lemma. There is an effective procedure assigning to each formula $\varphi(x_1 \ldots x_n)$ of $\mathscr{L}$ a formula $s_\varphi(yx_1 \ldots x_n)$ of $\mathscr{L}$ (where y is a variable not occurring in φ) such that for $\mathscr{M} \models T_{BAU}$, properties (i) and (ii) are equivalent and (ii) implies (iii):

(i) $\| \varphi [b_1 \ldots b_n] \|$ is clopen for every $\varphi(x_1 \ldots x_n)$ in $\mathscr{L}$ and $b_1, \ldots, b_n \in B$;

(ii) $\mathscr{M} \models \forall x_1 \ldots \forall x_n \exists y\, s_\varphi(yx_1 \ldots x_n)$ for every $\varphi(x_1 \ldots x_n)$ in $\mathscr{L}$;

(iii) if $b_1, \ldots, b_n \in B$, then $a = e(c)$ where $c = \| \varphi [b_1 \ldots b_n] \|$ is the unique element b of B such that $\mathscr{M} \models s_\varphi [bb_1 \ldots b_n]$.

Proof. The inductive definition of s_φ will show that (i) is equivalent to (ii) and (i) implies (iii), the interesting cases being φ atomic or φ existential. In both cases the fact that $\| \varphi [\ldots] \|$ is clopen will be expressed by stating "$a\ (= e(\| \varphi [\ldots] \|)$ is the largest element of A such that $e^{-1}(a) \subseteq \| \varphi [\ldots] \|$". This includes, if φ has the form $\exists x \psi$, the maximum principle for the Boolean valuation

$$\psi, b_1 \ldots b_n \to \| \psi [b_1 \ldots b_n] \|$$

of $\mathscr{M}$ in C: there is some $b \in B$ such that

$$\| \psi [b' b_1 \ldots b_n] \| \leqslant \| \psi [bb_1 \ldots b_n] \|$$

for every $b' \in B$, and hence $\| \psi [bb_1 \ldots b_n] \| = \| \exists x \psi [xb_1 \ldots b_n] \|$. We now proceed to define the formulas s_φ.

a) Suppose φ is an atomic formula of $\mathscr{L}_{BA}$, i.e. φ has the form $t_1(x_1 \dots x_n) = t_2(x_1 \dots x_n)$ where t_1, t_2 are terms in $\mathscr{L}_{BA}$. Let $s_\varphi(yx_1 \dots x_n)$ be the formula

$$U(y) \wedge y \cdot t_1 = y \cdot t_2 \wedge \forall y' (U(y') \wedge y' \cdot t_1 = y' \, t_2 \rightarrow y' \leqslant y) .$$

b) Suppose φ has the form $U(t(x_1 \dots x_n))$ where t is a term in $\mathscr{L}_{BA}$. Let ψ, χ be the atomic $\mathscr{L}_{BA}$-formulas "$t = 1$" resp. "$t = 0$". Let s_φ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)] .$$

c) Suppose φ has the form $\neg \psi(x_1 \dots x_n)$. Let s_φ be the formula

$$\exists y_1 [y = -y_1 \wedge s_\psi(y_1 x_1 \dots x_n)] .$$

d) Suppose φ has the form $\psi(x_1 \dots x_n) \vee \chi(x_1 \dots x_n)$. Let s_φ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)] .$$

e) Suppose φ has the form $\exists x \psi(x x_1 \dots x_n)$. Let s_φ be the formula

$$\exists x s_\psi(y x x_1 \dots x_n) \wedge \forall x' \forall y' [s_\psi(y' x' x_1 \dots x_n) \rightarrow y' \leqslant y] .$$

Let σ be the $\mathscr{L}_{BA}$-formula stating that the supremum of the atoms of a BA exists; σ^U is the relativization of σ to the one-place predicate U of $\mathscr{L}$. The models of $T_{BA} \cup \{\sigma\}$ are called separated BAs in [3]. Let T be the $\mathscr{L}$-theory

$$T = T_{BAU} \cup \{\forall x_1 \dots \forall x_n \exists y \, s_\varphi(y x_1 \dots x_n) \mid \varphi(x_1 \dots x_n) \text{ in } \mathscr{L}\} \cup \{\sigma^U, s_\sigma(1)\} .$$

The last two axioms of T express, for a model $\mathscr{M} = (B, A)$ of T_{BAU}, that A and each stalk B_p are separated BAs. Let **K** be the class of $\mathscr{L}$-structures $\mathscr{M} = (B, A)$ where B is a cBA and A is relatively complete in B. We shall prove in section 4 that T is an axiomatization of the first-order theory of **K**. The easy part of this is:

3.2. THEOREM. *Each structure $\mathscr{M}$ in* **K** *is a model of T.*

Proof. Let $\mathscr{M} = (B, A) \in \mathbf{K}$, i.e. B is complete and A is relatively complete in B. Hence $\mathscr{M} \models T_{BAU}$ and A is a separated BA. By 1.1, $\| \varphi[b_1 \dots b_n] \|$ is clopen for every atomic formula φ of $\mathscr{L}$ and arbitrary $b_1, \dots, b_n \in B$. If $\| \varphi[b_1 \dots b_n] \|$ and $\| [\psi[b_1 \dots b_n] \|$ are clopen subsets of X, so are $\| \neg \varphi[b_1 \dots b_n] \|$ and $\| (\varphi \vee \psi)[b_1 \dots b_n] \|$. Hence we assume that φ

has the form $\exists x \psi (xx_1 \dots x_n)$ and that $\| \psi [bb_1 \dots b_n] \|$ is clopen for fixed $b_1, \dots, b_n \in B$ and arbitrary $b \in B$. For the rest of the proof, we omit the parameters $b_1 \dots, b_n$. Let

$$u = \cup \{ \| \psi [\beta] \| \mid \beta \in B \} .$$

By our inductive assumption, u is an open subset of X. Choose, by Zorn's lemma, a maximal family $F = \{(b_i, c_i) \mid i \in I\}$ such that $b_i \in B$, c_i is a clopen subset of u, $c_i \subseteq \| \psi [b_i] \|$, $i \neq j$ implies $c_i \cap c_j = \phi$. It follows that c, the closure of $\bigcup_{i \in I} c_i$, includes u (by maximality of F). A is a cBA, hence X is extremally disconnected and c is clopen. By completeness of B, there is some $b \in B$ such that $b \cdot e(c_i) = b_i$ for $i \in I$. Thus, for $i \in I$, $c_i \subseteq \| \psi [b] \|$. So, for $\beta \in B$, $\| \psi [\beta] \| \subseteq u \subseteq c \subseteq \| \psi [b] \| = \| \exists x \psi (x) \|$.

Finally we show that B_p is separated for each $p \in X$. Let $\alpha (x)$ be the $\mathscr{L}_{BA}$-formula stating that x is an atom and let $\beta (x)$, $\gamma (x)$ be the $\mathscr{L}_{BA}$-formulas $\alpha (x) \vee x = 0$ resp. $\forall y (\alpha (y) \to y \leqslant x)$. Put $M = \{f \in B \mid \| \beta [f] \| = 1 \}$ and let b be the supremum of M in B. We show that $b(p)$ is, for each $p \in X$, the supremum of the atoms of B_p.

First suppose $s \in B_p$ is an atom of B_p. There is some $f \in M$ such that $f(p) = s$ (note that $\| \alpha [f] \|$ is clopen for each $f \in B$). So $f \leqslant b$ and $s = f(p) \leqslant b(p)$. — On the other hand, suppose $t \in B_p$ and $s \leqslant t$ for every atom s of B_p. Choose $g \in B$ such that $g(p) = t$. Then $p \in c = \| \gamma [g] \|$. For $f \in M$, $e(c) \cdot f \leqslant g$, since $q \in c$ implies that $f(q)$ is zero or an atom of B_q and thus $f(q) \leqslant g(q)$. By the definition of b, $e(c) \cdot b \leqslant g$. This implies (by $p \in c$) $b(p) \leqslant g(p) = t$.

4. DECIDABILITY AND COMPLETIONS OF $Th(\mathbf{K})$

Call $T_{sBA} = T_{BA} \cup \{\sigma\}$ the theory of separated BAs, where T_{BA} is the theory of BAs and σ was defined in section 3. We give a short review of the completions of T_{sBA}. Let, for $n \in \omega$, φ_n be the $\mathscr{L}_{BA}$-sentence stating that there are exactly n atoms and ψ the $\mathscr{L}_{BA}$-sentence stating that there is a non-zero atomless element. Let $\chi_n = \neg (\varphi_0 \vee \dots \vee \varphi_{n-1})$; so χ_n says that there are at least n atoms. Define, for $n \in \omega + 1$ and $i \in 2 = \{0, 1\}$, an $\mathscr{L}_{BA}$-theory T_{ni} by

$$T_{n0} = T_{sBA} \cup \{\varphi_n, \neg \psi\}$$
$$T_{n1} = T_{sBA} \cup \{\varphi_n, \psi\}$$

for $n \in \omega$, and

$$T_{\omega 0} = T_{sBA} \cup \{\chi_n \mid n \in \omega\} \cup \{\neg \psi\}$$
$$T_{\omega 1} = T_{sBA} \cup \{\chi_n \mid n \in \omega\} \cup \{\psi\}.$$

Put $\tau = \{T_{ni} \mid n \in \omega + 1, i \in 2\}$. It is then clear that each separated *BA* satisfies exactly one of the theories in τ, and for each $t \in \tau$ there is a *cBA* satisfying t. Moreover, any two models of any $t \in \tau$ are elementarily equivalent by 5.5.10 in [1]. Thus the theories $t \in \tau$ are just the completions of T_{sBA} and can be thought of as being the elementary equivalence types of separated *BAs* or *cBAs*. Moreover, an $\mathscr{L}_{BA}$-sentence holds in every separated *BA* iff it holds in every *cBA*. The following proposition is essential for the main theorems of this section:

4.1. PROPOSITION. *Let $s, t \in \tau$. Then there is a structure (B, A) in* **K** *such that A is a model of s and each stalk B_p is a model of t.*

Proof. By the above remarks, choose *cBAs* A and F which are models of s resp. t. Let $A * F$ be the free product of A and F. Thus A is relatively complete in $A * F$ and each stalk $(A * F)_p$, where p is an ultrafilter of A, is easily seen to be isomorphic to F, hence a model of t. Unfortunately, $A * F$ is incomplete unless A or F is finite. So let $B = (A * F)^*$ be the completion of $A * F$; note that $A * F$ is a dense subalgebra of B. $(B, A) \in \mathbf{K}$, since the inclusion maps from A to $A * F$ and from $A * F$ to B are complete. For $p \in X$ (the Stone space of A), B_p is a separated *BA* by 3.2 but in general a proper extension of $(A * F)_p$. We show, with the notations of section 1, that B_p is elementarily equivalent to F. For the following proof of this, recall that, for $f \in F \setminus \{0\}$ and $p \in X$, $\pi_p(f) = f(p) \neq 0$ since F is independent from A in $A * F \subseteq B$. Thus, the restriction of $\pi_p : B \to B_p$ to F is a monomorphism. The elementary equivalence of B_p and F is established by the following four claims.

Claim 1. For each atom f of F, $f(p)$ is an atom of B_p (hence, if F has at least n atoms, where $n \in \omega$, then B_p has at least n atoms): clearly, $f(p) > 0$ for $p \in X$. Assume that

$$u = \{p \in X \mid f(p) \text{ is not an atom of } B_p\}$$

is non-empty. By 3.2, u is a clopen subset of X. Choose, by the maximum principle stated in section 3, $b \in B$ such that $b(p) = 0$ for $p \notin u$ and $0 < b(p) < f(p)$ for $p \in u$. Since $b > 0$, choose $a \in A$ and $g \in F$ such that $0 < a \cdot g \leqslant b$; let $p \in X$ such that $a(p) \cdot g(p) \neq 0$. Thus $p \in u$, $a(p) = 1$, and

$0 < g(p) \leqslant b(p) < f(p)$. It follows that $0 < g < f$, contradicting the fact that f was an atom of F.

Claim 2. If B_p has at least n atoms, where $1 \leqslant n < \omega$, then F has at least n atoms: assume that M is a subset of $At(B_p)$, the set of atoms of B_p, such that M has exactly n elements but $At(F)$ has at most $n-1$ elements. We prove:

(a) Let $x \in M$. Then there is $f_x \in At(F)$ such that $f_x(p) = x$.

Claim 2 follows from *(a)*, since the assignment of f_x to x is injective. And *(a)* will follow from

(b) Let $x \in M$, u a clopen neighbourhood of p such that, w.l.o.g., for $q \in u$, B_q has at least one atom. Let $b \in B$ such that, for $q \notin u$, $b(q) = 0$ and for $q \in u$, $b(q)$ is an atom of B_q, and $b(p) = x$. Then there are $q \in u$ and $f \in At(F)$ such that $f(q) = b(q)$. (Hence $At(F)$ is non-empty).

Proof of *(b)*. By $b > 0$, choose $a \in A$, $f \in F$ such that $0 < a \cdot f \leqslant b$. Since $b(q) = 0$ for $q \notin u$, there is some $q \in u$ such that $a(q) \cdot f(q) \neq 0$, which implies $0 < f(q) \leqslant b(q) \,.\, f(q) = b(q)$, since $b(q)$ is an atom of B_q. Finally $f \in At(F)$, since a splitting of f in F into two non-zero disjoint elements would give rise to a splitting of $b(q)$ in B_q.

Proof of *(a)*. Let $x \in M$ and choose u and b as in *(b)*. Assume *(a)* is false. Then, for each $f \in At(F)$, $f(p) \neq x = b(p)$; by finiteness of $At(F)$, there is a clopen neighbourhood v of p such that, for $q \in v$ and $f \in At(F)$, $b(q) \neq f(q)$. Let $c \in B$ such that $c(q) = 0$ for $q \notin v$ and $c(q) = b(q)$ for $q \in v$. This contradicts *(b)*, applied to v and c instead of u and b.

Claim 3. If F has a non-zero atomless element f (which means that $F \restriction f$ is atomless), then each B_p has a non-zero atomless element x: let $x = \pi_p(f)$. $x > 0$, since π_p is one-one on F. $F \restriction f$ and hence, by Claim 2, $(B \restriction f)_p$ is atomless. So $B_p \restriction x = \pi_p(B \restriction f) = (B \restriction f)_p$ is atomless.

Claim 4. If B_p has a non-zero atomless element x, then F has a non-zero atomless element f: assume that F is atomic. Let

$$u = \{q \in X \mid B_q \text{ is not atomic}\}.$$

u is a clopen neighbourhood of p. By the maximum principle, choose $b \in B$ such that $b(q) = 0$ for $q \notin u$, $b(q)$ is a non-zero atomless element of

B_q for $q \in u$, $b(p) = x$. Choose $a \in A$, $g \in F$ such that $0 < a \cdot g \leqslant b$; w.l.o.g., g is an atom of F. Choose $q \in X$ such that $a(q) \cdot g(q) \neq 0$. Thus $q \in u$ and $g(q) \leqslant b(q)$. By Claim 1, $g(q)$ is an atom of B_q, contradicting the choice of $b(q)$.

4.2. REMARK. Suppose that, for every i in an index set I, $\mathcal{M}_i = (B_i, A_i)$ is an element of **K**. Then $\mathcal{M} = (B, A)$, where $B = \prod_{i \in I} B_i$ and $A = \prod_{i \in I} A_i$, is in **K**. Let $\varphi(x_1 \dots x_k)$ be an $\mathcal{L}$-formula and $b_1, \dots, b_k \in B$, $b_j = (b_{ij})_{i \in I}$. Put $a_i = e(\| \varphi[b_{i1} \dots b_{ik}] \|^{\mathcal{M}_i})$. Then

$$e(\| \varphi[b_1 \dots b_k] \|^{\mathcal{M}}) = (a_i)_{i \in I}.$$

Proof. By induction on the complexity of φ.

We shall need the following Feferman-Vaught theorem about sheaves over Boolean spaces from [2]:

4.3. THEOREM (Comer). *Let $\mathcal{L}$ be an arbitrary language. There is an effective assignment*

$$\varphi(x_1 \dots x_k) \mapsto (\Phi; \vartheta_1, \dots, \vartheta_m)$$

for $\mathcal{L}$-formulas $\varphi(x_1 \dots x_k)$ such that

a) $\vartheta_1, \dots, \vartheta_m$ *are $\mathcal{L}$-formulas having at most the free variables $x_1 \dots x_k$, and*

$$\models \Big(\bigvee_{1 \leqq i \leqq m} \vartheta_i \Big) \wedge \bigwedge_{1 \leqq i < j \leqq m} \neg (\vartheta_i \wedge \vartheta_j)$$

b) Φ *is an $\mathcal{L}_{BA}$-formula having at most the free variables $y_1 \dots y_m$;*

c) *for each sheaf $\mathcal{S} = (S, \pi, X. \mu)$ of $\mathcal{L}$-structures such that X is a Boolean space and $\| \psi[f_1 \dots f_n] \|$ is a clopen subset of X for every $\psi(x_1 \dots x_n)$ in $\mathcal{L}$ and $f_1, \dots, f_n \in \Gamma(\mathcal{S})$: if $b_1, \dots, b_k \in \Gamma(\mathcal{S})$, then*

$$\Gamma(\mathcal{S}) \models \varphi[b_1 \dots b_k] \quad \textit{iff} \quad C \models \Phi[c_1 \dots c_m],$$

where C is the BA of clopen subsets of X and $c_i = \| \vartheta_i [b_1 \dots b_k] \|$.

For two separated *BAs* A and A', let I be the set of partial functions f from A to A' such that $\mathrm{dom}(f) = \{a_1, \dots, a_n\}$ is a finite partition of A (where some of the a_i may be zero), $rge(f) = \{a_1', \dots, a_n'\}$ where $a_i' = f(a_i)$ is a partition of A', and every $A \restriction a_i$ is elementarily equivalent to $A' \not\vdash a_i'$. If A, A' are $\aleph_1$-saturated or σ-complete, the following conditions are equivalent:

a) $A \equiv A'$;

b) I is non-empty;

c) I has the back-and-forth property.

Moreover, if $f \in I$ is as above and A, A' are arbitrary separated *BAs*, then $(A, a_1, ..., a_n) \equiv (A', a_1', ..., a_n')$.

Let T_{sBA2} be the $\mathscr{L}$-theory

$$T_{sBA2} = T_{sBA} \cup \{\forall x (U(x) \leftrightarrow x = 0 \vee x = 1)\} .$$

Since T_{BA} is decidable, T_{sBA} and T_{sBA2} are decidable.

4.4. THEOREM. *There is an effective procedure deciding for every $\mathscr{L}$-sentence φ whether $T \vdash \varphi$. Moreover, $T \vdash \varphi$ if and only if φ holds in every model $\mathscr{M}$ in* **K**.

Proof. Let φ be given. Construct $(\Phi(y_1 ... y_m); \vartheta_1, ..., \vartheta_m)$ by 4.3. For every i such that $1 \leqslant i \leqslant m$, decide whether $T_{sBA2} \vdash \neg \vartheta_i$. W.l.o.g., assume that $T_{sBA2} \cup \{\vartheta_i\}$ is consistent for $1 \leqslant i \leqslant r$ and inconsistent for $r + 1 \leqslant i \leqslant m$. By $\vdash \vartheta_1 \vee ... \vee \vartheta_m$, we have $1 \leqslant r$ (it is possible that $r = m$). Next, construct the formula

$$\Phi'(y_1 ... y_m) = \Big(\bigwedge_{r+1 \leqq i \leqq m} (y_i = 0) \rightarrow \Phi(y_1 ... y_m)\Big) .$$

We show the equivalence of

a) $T \vdash \varphi$;

b) $\mathscr{M} \models \varphi$ for every $\mathscr{M} \in \mathbf{K}$;

c) $T_{sBA} \vdash \forall y_1 ... \forall y_m \, \Phi'(y_1 ... y_m)$.

Then, by decidability of T_{sBA}, T is decidable and 4.4 is proved. *a)* implies *b)* by 3.2. To prove that *c)* implies *a)*, assume there is $\mathscr{M} \models T$ such that $\mathscr{M} \not\models \varphi$, e.g. $\mathscr{M} = (B, A)$. Put $a_i = e(\| \vartheta_i \|^{\mathscr{M}})$. By 4.3 and $\mathscr{M} \not\models \varphi$, we see $A \not\models \Phi[a_1 ... a_m]$. By our choice of $r \leqslant m$, we get $a_{r+1} = ... = a_m = 0$. Thus $A \not\models \Phi'[a_1 ... a_m]$ and c) is false. Now assume c) does not hold; we show that b) is false. Let A' be a separated *BA* and $a_1', ..., a_m' \in A'$ such that $a_{r+1}' = ... = a_m' = 0$ and $A' \not\models \Phi[a_1' ... a_m']$. W.l.o.g., $a_i' \neq 0$ for $1 \leqslant i \leqslant r$. By choice of r, there are $t_1, ..., t_r \in \tau$ such that $t_i \models \vartheta_i$ for $1 \leqslant i \leqslant r$.

Let, for these i, s_i be the element of τ such that $A' \restriction a_i' \models s_i$. By 4.1, there are $\mathcal{M} = (B, A) \in \mathbf{K}$ and $a_1, \ldots, a_r \in A$ such that $1 = a_1 + \ldots + a_r$, $a_i \cdot a_j = 0$ for $1 \leqslant i < j \leqslant r$, $A \restriction a_i \models s_i$ and $(B \restriction a_i)_p \models t_i$ for those $p \in X$ satisfying $a_i(p) = 1$. So $e(\| \vartheta_i \|^{\mathcal{M}}) = a_i$ by 4.2. Put $a_{r+1} = \ldots = a_m = 0$. It follows that $(A, a_1, \ldots, a_m) \equiv (A', a_1', \ldots, a_m')$, $A \not\models \Phi[a_1 \ldots a_m]$ and $\mathcal{M} \not\models \varphi$ by 4.3.

In the next theorem, we characterize elementary equivalence of models of T. Call the following sentences in $\mathcal{L}_{BA}$ basic sentences: $\varphi_n \wedge \psi$, $\varphi_n \wedge \neg\psi$, $\chi_n \wedge \psi$, $\chi_n \wedge \neg\psi$ (where $n \in \omega$). It follows by the analysis of the completions of T_{sBA} given in the beginning of this section that for each $\mathcal{L}_{BA}$-sentence ϑ there are basic sentences $\beta_1, \ldots, \beta_n$ such that

$$T_{sBA} \vdash (\vartheta \leftrightarrow \bigvee_{i=1}^{n} \beta_i) \wedge \bigwedge_{1 \leq i < j \leq n} \neg(\beta_i \wedge \beta_j) .$$

This fact is easily extended to T_{sBA2}: by replacing each atomic formula $U(t)$ where t is a term in $\mathcal{L}_{BA}$ by "$t = 0 \vee t = 1$", we see that for each $\mathcal{L}$-sentence ϑ there are basic sentences $\beta_1, \ldots, \beta_n$ satisfying

$$T_{sBA2} \vdash (\vartheta \leftrightarrow \bigvee_{i=1}^{n} \ ) \wedge \bigwedge_{1 \leq i < j \leq n} \neg(\beta_i \wedge \beta_j) .$$

Now, if β, γ are basic sentences, let $\sigma_{\beta\gamma}$ be the following $\mathcal{L}$-sentence:

$$\sigma_{\beta\gamma} = \exists y (\gamma^y \wedge s_\beta(y)),$$

where $s_\beta(y)$ is the $\mathcal{L}$-formula assigned to β in 3.1 and γ^y is the result of relativizing the quantifiers $\exists x\varphi \ldots$ in γ to $\exists x (U(x) \wedge x \leqslant y \wedge \varphi^y \ldots)$. A model (B, A) of T satisfies $\sigma_{\beta\gamma}$ iff $A \restriction a \models \gamma$, where $a = e(c)$ and $c = \| \beta \|$.

4.5. THEOREM. *Let $\mathcal{M} = (B, A)$, $\mathcal{M}' = (B', A')$ be models of T. Then $\mathcal{M}$ is elementarily equivalent to $\mathcal{M}'$ if and only if, for any basic sentences β, γ,*

$$\mathcal{M} \models \sigma_{\beta\gamma} \quad \text{iff} \quad \mathcal{M}' \models \sigma_{\beta\gamma} .$$

Proof. The only-if-part is clear. Suppose that $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same sentences of the form $\sigma_{\beta\gamma}$. Let φ be an $\mathcal{L}$-sentence and $\mathcal{M} \models \varphi$; we want to show that $\mathcal{M}' \models \varphi$. Let $(\Phi(y_1 \ldots y_m); \vartheta_1, \ldots, \vartheta_m)$ be the sequence assigned to φ by 4.3; every ϑ_i is an $\mathcal{L}$-sentence. Put $a_i = e(\| \vartheta_i \|^{\mathcal{M}})$; by 4.3 and $e: C \to A$ being an isomorphism, we have that $\{a_1, \ldots, a_m\}$

is a partition of A and $A \models \Phi\,[a_1 \ldots a_m]$. In the same way, put $a_i' = e'(\|\vartheta_i\|^{\mathcal{M}'})$; $\{a_1', \ldots, a_m'\}$ is a partition of A'. It is sufficient to show that $(A, a_1, \ldots, a_m) \equiv (A', a_1', \ldots, a_m')$, for this implies $A' \models \Phi\,[a_1' \ldots a_m']$ and finally $\mathcal{M}' \models \varphi$ by 4.3.

For every ϑ_i, choose basic sentences $\beta_{i1}, \ldots, \beta_{in_i}$ such that

$$T_{sBA2} \vdash (\vartheta_i \leftrightarrow \bigvee_j \beta_{ij}) \wedge \bigwedge_{j<l} \neg\, (\beta_{ij} \wedge \beta_{il})\,.$$

Put $\alpha_{ij} = e(\|\beta_{ij}\|^{\mathcal{M}})$, $\alpha_{ij}' = e'(\|\beta_{ij}\|^{\mathcal{M}'})$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n_i$. Then a_i is the disjoint sum of the α_{ij} $(1 \leqslant j \leqslant n_i)$, a_i' is the disjoint sum of the α'_{ij} $(1 \leqslant j \leqslant n_i)$. For every i, j,

$$A \restriction \alpha_{ij} \equiv A' \restriction \alpha_{ij}' :$$

let γ be any basic sentence of $\mathcal{L}_{BA}$ and assume $A \restriction \alpha_{ij} \models \gamma$; we want to show that $A' \restriction \alpha_{ij}' \models \gamma$. But $A \restriction \alpha_{ij} \models \gamma$ means that $\mathcal{M} \models \sigma_{\beta_{ij}\gamma}$. By our main assumption, $\mathcal{M}' \models \sigma_{\beta_{ij}\gamma}$ and $A' \restriction \alpha'_{ij} \models \gamma$.

We have shown that the partial function f mapping α_{ij} to α_{ij}' is an element of the set of back-and-forth-isomorphisms defined after 4.3. Hence,

$$(A, \alpha_{11}, \ldots, \alpha_{mn_m}) \equiv (A', \alpha_{11}', \ldots, \alpha_{mn_m}')$$

and

$$(A, a_1, \ldots, a_m) \equiv (A', a_1', \ldots, a_m')\,.$$

We shall finally describe the completions of T by giving a one-one correspondance between a set P (consisting of pairs of mappings from $\omega \times 2$ to $(\omega+1) \times 2$) and these completions. For $m, m' \in \omega + 1$ and $j, j' \in 2$, define

$$(m,j) + (m',j') = (m'',j'')$$

where m'' is the cardinal sum of m and m' and j'' is the maximum of j and j'. Let

$$P = \{(\alpha, \rho) \mid \alpha, \rho : \omega \times 2 \to (\omega+1) \times 2 \text{ and, for } (n, i) \in \omega \times 2,\ \rho\,(n, i) = \rho\,(n+1, i) + \alpha\,(n, i)\}\,.$$

In the following definition, we refer to the $\mathcal{L}_{BA}$-theories T_{ni} defined in the beginning of this section. For $(\alpha, \rho) \in P$, let $T_{\alpha\rho}$ the $\mathcal{L}$-theory

$$\begin{aligned} T_{\alpha\rho} = T &\cup \{\exists x\,(\sigma_{(\varphi_n \wedge \neg\psi)}(x) \wedge \gamma^x) \mid n \in \omega\,,\ \gamma \in T_{\alpha(n,0)}\} \\ &\cup \{\exists x\,(\sigma_{(\chi_n \wedge \neg\psi)}(x) \wedge \gamma^x) \mid n \in \omega\,,\ \gamma \in T_{\rho(n,0)}\} \\ &\cup \{\exists x\,(\sigma_{(\varphi_n \wedge \psi)}(x) \wedge \gamma^x) \mid n \in \omega\,,\ \gamma \in T_{\alpha(n,1)}\} \\ &\cup \{\exists x\,(\sigma_{(\chi_n \wedge \psi)}(x) \wedge \gamma^x) \mid n \in \omega\,,\ \gamma \in T_{\rho(n,1)}\}\,. \end{aligned}$$

If $\mathcal{M} = (B, A)$ is a model of T, then $\mathcal{M} \models T_{\alpha\rho}$ iff, for $a_1 = e(\| \varphi_n \wedge \neg \psi \|^{\mathcal{M}})$ $A \upharpoonright a_1 \models T_{\alpha(n,0)}, \ldots,$ for $a_4 = e(\| \chi_n \wedge \psi \|^{\mathcal{M}})$, $A \upharpoonright a_4 \models T_{\rho(n,1)}$.

4.6. Theorem. $\{T_{\alpha\rho} \mid (\alpha, \rho) \in P\}$ *is the set of completions of* T. *Moreover, each* $T_{\alpha\rho}$ *has a model in* **K**.

Proof. If (α, ρ) and (α', ρ') are different elements of P, then $T_{\alpha\rho} \cup T_{\alpha'\rho'}$ is inconsistent (recall that every T_{mj}, where $m \in \omega + 1$, $j \in 2$, is complete in $\mathcal{L}_{BA}$). If $\mathcal{M}$ is a model of T, there is some $(\alpha, \rho) \in P$ such that $\mathcal{M} \models T_{\alpha\rho}$ (e.g., put $a_1 = e(\| \varphi_n \wedge \neg \psi \|^{\mathcal{M}})$ and let $\alpha(n, 0)$ be the pair $(k, j) \in (\omega + 1) \times 2$ such that $A \upharpoonright a_1 \models T_{kj}$, etc.). If $(\alpha, \rho) \in P$ and $\mathcal{M}$, $\mathcal{M}'$ are models of $T_{\alpha\rho}$, then $\mathcal{M}$ and $\mathcal{M}'$ are elementarily equivalent by 4.5, since $T_{\alpha\rho}$ says which sentences of the form $\sigma_{\beta\gamma}$ are satisfied in $\mathcal{M}$ and $\mathcal{M}'$. So it is sufficient to prove that each $T_{\alpha\rho}$ has a model which lies even in **K**.

For simplicity, we construct $\mathcal{M} \in \mathbf{K}$ satisfying the part of $T_{\alpha\rho}$ which refers to $T_{\alpha(n,0)}$ and $T_{\rho(n,0)}$ — for, if $\mathcal{N} \in \mathbf{K}$ satisfies the part of $T_{\alpha\rho}$ which refers to $T_{\alpha(n,1)}$ and $T_{\rho(n,1)}$, then $\mathcal{M} \times \mathcal{N} \in \mathbf{K}$ is a model of $T_{\alpha\rho}$. Abbreviate $\alpha(n, 0)$ by t_n, $\rho(n, 0)$ by s_n. We first construct a complete *BA* A and a sequence $(a_n)_{n \in \omega}$ in A such that the a_n are pairwise disjoint and

$$(*) \quad A \upharpoonright a_n \models t_n, \qquad A \upharpoonright r_n \models s_n$$

where $r_n = -(a_0 + \ldots + a_{n-1})$. Let A be a complete *BA* which is a model of s_0. Suppose $a_0, \ldots, a_{n-1} \in A$ are pairwise disjoint and $a_0, \ldots, a_{n-1}$, r_n satisfy (*). Since $s_n = s_{n+1} + t_n$, $A \upharpoonright r_n \models s_n$ and A is complete, there are a_n and $r_{n+1} \in A$ such that $r_n = a_n + r_{n+1}$, $a_n \cdot r_{n+1} = 0$, $A \upharpoonright a_n \models t_n$ and $A \upharpoonright r_{n+1} \models s_{n+1}$. — Finally, let $a_\omega = -\sum_{n \in \omega} a_n$. By the proof of 4.1, there is, for $n \in \omega$, $\mathcal{M}_n = (B_n, A_n) \in \mathbf{K}$ such that $A_n = A \upharpoonright a_n$ and each stalk $(B_n)_p$ of the sheaf representation of $\mathcal{M}_n$ is a model of $\varphi_n \wedge \neg \psi$. Moreover there is $\mathcal{M}_\omega = (B_\omega, A_\omega) \in \mathbf{K}$ such that $A_\omega = A \upharpoonright a_\omega$ and each stalk $(B_\omega)_p$ of the sheaf representation of $\mathcal{M}_\omega$ is a model of $T_{\omega 0}$. Put $\mathcal{M} = (B, A)$ where B is a complete *BA* which lies over A as $\prod_{n \in \omega} B_n$ lies over $\prod_{n \in \omega} A_n$. By 4.2, $e(\| \varphi_n \wedge \neg \psi \|^{\mathcal{M}}) = a_n$ and $e(\| \chi_n \wedge \neg \psi \|^{\mathcal{M}}) = r_n$;so $\mathcal{M}$ is a model of the part of $T_{\alpha\rho}$ referring to $T_{\alpha(n,0)}$ and $T_{\rho(n,0)}$.

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