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3. Truth values in A for statements about (B, A)

For the rest of this paper, let $\mathscr{L}_{BA} = \{+, \cdot, -, 0, 1\}$ the language of BAs and $\mathscr{L} = \mathscr{L}_{BA} \cup \{U\}$. Let T_{BAU} be the theory in $\mathscr{L}$ such that the models of T_{BAU} have the form $(B, +, \cdot, -, 0, 1, A)$ where $(B, \ldots)$ is a BA and A is a subalgebra of B. We abbreviate a model $(B, \ldots, A)$ of T_{BAU} by $\mathscr{M} = (B, A)$. We assume the construction and notations of section 1. For each $\mathscr{L}$-formula $\varphi(x_1 \ldots x_n)$ and $b_1, \ldots, b_n \in B$, we defined

$$\| \varphi [b_1 \ldots b_n] \| = \{ p \in X \mid B_p \models \varphi [b_1(p) \ldots b_n(p)] \}$$

where B_p abbreviates $(B_p, 2)$ and 2 is the two-element BA. Our first claim is that if $c = \| \varphi [b_1 \ldots b_n] \|$ is a clopen subset of X for every φ, then $e(c) \in A$ is first-order definable in $\mathscr{M} = (B, A)$ from the parameters $b_1, \ldots, b_n \in B$:

3.1. Lemma. There is an effective procedure assigning to each formula $\varphi(x_1 \ldots x_n)$ of $\mathscr{L}$ a formula $s_\varphi(yx_1 \ldots x_n)$ of $\mathscr{L}$ (where y is a variable not occurring in φ) such that for $\mathscr{M} \models T_{BAU}$, properties (i) and (ii) are equivalent and (ii) implies (iii):

(i) $\| \varphi [b_1 \ldots b_n] \|$ is clopen for every $\varphi(x_1 \ldots x_n)$ in $\mathscr{L}$ and $b_1, \ldots, b_n \in B$;

(ii) $\mathscr{M} \models \forall x_1 \ldots \forall x_n \exists y\, s_\varphi(yx_1 \ldots x_n)$ for every $\varphi(x_1 \ldots x_n)$ in $\mathscr{L}$;

(iii) if $b_1, \ldots, b_n \in B$, then $a = e(c)$ where $c = \| \varphi [b_1 \ldots b_n] \|$ is the unique element b of B such that $\mathscr{M} \models s_\varphi [bb_1 \ldots b_n]$.

Proof. The inductive definition of s_φ will show that (i) is equivalent to (ii) and (i) implies (iii), the interesting cases being φ atomic or φ existential. In both cases the fact that $\| \varphi [\ldots] \|$ is clopen will be expressed by stating "$a\,(= e(\| \varphi [\ldots] \|)$ is the largest element of A such that $e^{-1}(a) \subseteq \| \varphi [\ldots] \|$". This includes, if φ has the form $\exists x \psi$, the maximum principle for the Boolean valuation

$$\psi, b_1 \ldots b_n \to \| \psi [b_1 \ldots b_n] \|$$

of $\mathscr{M}$ in C: there is some $b \in B$ such that

$$\| \psi [b' b_1 \ldots b_n] \| \leqslant \| \psi [bb_1 \ldots b_n] \|$$

for every $b' \in B$, and hence $\| \psi [bb_1 \ldots b_n] \| = \| \exists x \psi [xb_1 \ldots b_n] \|$. We now proceed to define the formulas s_φ.

a) Suppose φ is an atomic formula of $\mathscr{L}_{BA}$, i.e. φ has the form $t_1(x_1 \dots x_n) = t_2(x_1 \dots x_n)$ where t_1, t_2 are terms in $\mathscr{L}_{BA}$. Let $s_\varphi(yx_1 \dots x_n)$ be the formula

$$U(y) \wedge y \cdot t_1 = y \cdot t_2 \wedge \forall y' (U(y') \wedge y' \cdot t_1 = y' \, t_2 \to y' \leqslant y) .$$

b) Suppose φ has the form $U(t(x_1 \dots x_n))$ where t is a term in $\mathscr{L}_{BA}$. Let ψ, χ be the atomic $\mathscr{L}_{BA}$-formulas "$t = 1$" resp. "$t = 0$". Let s_φ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1x_1 \dots x_n) \wedge s_\chi(y_2x_1 \dots x_n)] .$$

c) Suppose φ has the form $\neg \psi(x_1 \dots x_n)$. Let s_φ be the formula

$$\exists y_1 [y = -y_1 \wedge s_\psi(y_1x_1 \dots x_n)] .$$

d) Suppose φ has the form $\psi(x_1 \dots x_n) \vee \chi(x_1 \dots x_n)$. Let s_φ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1x_1 \dots x_n) \wedge s_\chi(y_2x_1 \dots x_n)] .$$

e) Suppose φ has the form $\exists x \psi(xx_1 \dots x_n)$. Let s_φ be the formula

$$\exists x s_\psi(yxx_1 \dots x_n) \wedge \forall x' \forall y' [s_\psi(y'x'x_1 \dots x_n) \to y' \leqslant y] .$$

Let σ be the $\mathscr{L}_{BA}$-formula stating that the supremum of the atoms of a *BA* exists; σ^U is the relativization of σ to the one-place predicate U of $\mathscr{L}$. The models of $T_{BA} \cup \{\sigma\}$ are called separated *BA*s in [3]. Let T be the $\mathscr{L}$-theory

$$T = T_{BAU} \cup \{\forall x_1 \dots \forall x_n \exists y \, s_\varphi(yx_1 \dots x_n) \mid \varphi(x_1 \dots x_n) \text{ in } \mathscr{L}\} \cup \{\sigma^U, s_\sigma(1)\} .$$

The last two axioms of T express, for a model $\mathscr{M} = (B, A)$ of T_{BAU}, that A and each stalk B_p are separated *BA*s. Let **K** be the class of $\mathscr{L}$-structures $\mathscr{M} = (B, A)$ where B is a *cBA* and A is relatively complete in B. We shall prove in section 4 that T is an axiomatization of the first-order theory of **K**. The easy part of this is:

3.2. THEOREM. *Each structure $\mathscr{M}$ in* **K** *is a model of T.*

Proof. Let $\mathscr{M} = (B, A) \in \mathbf{K}$, i.e. B is complete and A is relatively complete in B. Hence $\mathscr{M} \models T_{BAU}$ and A is a separated *BA*. By 1.1, $\| \varphi[b_1 \dots b_n] \|$ is clopen for every atomic formula φ of $\mathscr{L}$ and arbitrary $b_1, \dots, b_n \in B$. If $\| \varphi[b_1 \dots b_n] \|$ and $\| [\psi[b_1 \dots b_n] \|$ are clopen subsets of X, so are $\| \neg \varphi[b_1 \dots b_n] \|$ and $\| (\varphi \vee \psi)[b_1 \dots b_n] \|$. Hence we assume that φ

has the form $\exists x\, \psi\, (xx_1 \ldots x_n)$ and that $\| \psi\, [bb_1 \ldots b_n] \|$ is clopen for fixed $b_1, \ldots, b_n \in B$ and arbitrary $b \in B$. For the rest of the proof, we omit the parameters $b_1 \ldots, b_n$. Let

$$u = \cup \left\{ \| \psi\, [\beta] \| \,\middle|\, \beta \in B \right\}.$$

By our inductive assumption, u is an open subset of X. Choose, by Zorn's lemma, a maximal family $F = \left\{ (b_i, c_i) \,\middle|\, i \in I \right\}$ such that $b_i \in B$, c_i is a clopen subset of u, $c_i \subseteq \| \psi\, [b_i] \|$, $i \neq j$ implies $c_i \cap c_j = \phi$. It follows that c, the closure of $\bigcup_{i \in I} c_i$, includes u (by maximality of F). A is a cBA, hence X is extremally disconnected and c is clopen. By completeness of B, there is some $b \in B$ such that $b \cdot e\,(c_i) = b_i$ for $i \in I$. Thus, for $i \in I$, $c_i \subseteq \| \psi\, [b] \|$. So, for $\beta \in B$, $\| \psi\, [\beta] \| \subseteq u \subseteq c \subseteq \| \psi\, [b] \| = \| \exists x \psi\, (x) \|$.

Finally we show that B_p is separated for each $p \in X$. Let $\alpha\,(x)$ be the $\mathscr{L}_{BA}$-formula stating that x is an atom and let $\beta\,(x)$, $\gamma\,(x)$ be the $\mathscr{L}_{BA}$-formulas $\alpha\,(x) \vee x = 0$ resp. $\forall y\, (\alpha\,(y) \rightarrow y \leqslant x)$. Put $M = \{ f \in B \mid \| \beta\, [f] \| = 1 \}$ and let b be the supremum of M in B. We show that $b\,(p)$ is, for each $p \in X$, the supremum of the atoms of B_p.

First suppose $s \in B_p$ is an atom of B_p. There is some $f \in M$ such that $f\,(p) = s$ (note that $\| \alpha\, [f] \|$ is clopen for each $f \in B$). So $f \leqslant b$ and $s = f\,(p) \leqslant b\,(p)$. — On the other hand, suppose $t \in B_p$ and $s \leqslant t$ for every atom s of B_p. Choose $g \in B$ such that $g\,(p) = t$. Then $p \in c = \| \gamma\, [g] \|$. For $f \in M$, $e\,(c) \cdot f \leqslant g$, since $q \in c$ implies that $f\,(q)$ is zero or an atom of B_q and thus $f\,(q) \leqslant g\,(q)$. By the definition of b, $e\,(c) \cdot b \leqslant g$. This implies (by $p \in c$) $b\,(p) \leqslant g\,(p) = t$.

4. Decidability and completions of $Th\,(\mathbf{K})$

Call $T_{sBA} = T_{BA} \cup \{\sigma\}$ the theory of separated BAs, where T_{BA} is the theory of BAs and σ was defined in section 3. We give a short review of the completions of T_{sBA}. Let, for $n \in \omega$, φ_n be the $\mathscr{L}_{BA}$-sentence stating that there are exactly n atoms and ψ the $\mathscr{L}_{BA}$-sentence stating that there is a non-zero atomless element. Let $\chi_n = \neg\, (\varphi_0 \vee \ldots \vee \varphi_{n-1})$; so χ_n says that there are at least n atoms. Define, for $n \in \omega + 1$ and $i \in 2 = \{0, 1\}$, an $\mathscr{L}_{BA}$-theory T_{ni} by

$$T_{n0} = T_{sBA} \cup \{ \varphi_n, \neg\, \psi \}$$
$$T_{n1} = T_{sBA} \cup \{ \varphi_n, \psi \}$$