

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 32 (1986)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON CONSECUTIVE VALUES OF THE LIOUVILLE FUNCTION  
**Autor:** Hildebrand, Adolf  
**Kapitel:** 2. A Lemma  
**DOI:** <https://doi.org/10.5169/seals-55088>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 01.04.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

desired result. The drawback of this method is that it gives no indication on how to settle the general case of the conjecture, or even the case  $k = 4$ . It seems that for this completely new ideas are needed, and Chowla's remark on the difficulty of the problem appears to be justified, as far as the general form of the conjecture is concerned.

## 2. A LEMMA

LEMMA. *Each of the equations*

$$\lambda(15n-1) = \lambda(15n+1) = 1$$

and

$$\lambda(15n-1) = \lambda(15n+1) = -1$$

holds for infinitely many positive integers  $n$ .

*Proof.* Given a positive integer  $n_0 \geq 2$ , define  $n_i, i \geq 1$ , inductively by

$$n_{i+1} = n_i(4n_i^2 - 3) \quad (i \geq 0).$$

It is easily checked that

$$n_{i+1} \pm 1 = (n_i \pm 1)(2n_i \pm 1)^2 \quad (i \geq 0),$$

so that

$$\lambda(n_{i+1} \pm 1) = \lambda(n_i \pm 1) = \dots = \lambda(n_0 \pm 1) \quad (i \geq 0).$$

Also, it follows by induction that  $n_0 | n_i$  for all  $i \geq 0$ . Therefore, taking in turn  $n_0 = 15$  and  $n_0 = 30$  and noting that

$$\lambda(14) = \lambda(16) = 1, \quad \lambda(29) = \lambda(31) = -1,$$

we obtain two infinite sequences  $(n_i^{(+)})$  and  $(n_i^{(-)})$  with the required properties

$$n_i^{(\pm)} \equiv 0 \pmod{15}, \quad \lambda(n_i^{(+)} \pm 1) = 1, \quad \lambda(n_i^{(-)} \pm 1) = -1.$$

*Remark.* The same argument shows that for any completely multiplicative function  $f$  assuming only the values  $\pm 1$  and for given  $\varepsilon_1, \varepsilon_2 = \pm 1$  and  $a \geq 2$ , the system

$$n \equiv 0 \pmod{a}, \quad f(n-1) = \varepsilon_1, \quad f(n+1) = \varepsilon_2$$

has infinitely many solutions, provided it has at least one solution. It would be interesting to have an analogous result for three (or more) consecutive values, but the above method does not work in this case.

### 3. PROOF OF THE THEOREM, BEGINNING

We shall show here that each of the equations

$$(2) \quad \lambda(n) = \lambda(n+1) = \lambda(n-1) = 1$$

and

$$(2)' \quad \lambda(n) = \lambda(n+1) = \lambda(n-1) = -1$$

has infinitely many solutions. Since the arguments for the two cases are completely symmetric, we shall carry out the proof only in the case of equation (2).

Call an integer  $n \geq 2$  "good", if (2) holds for this  $n$ . We have to show that there are infinitely many good integers. To this end we shall show that for any positive integer  $n$  satisfying

$$(3) \quad n \equiv 0 \pmod{15}, \quad \lambda(n+1) = \lambda(n-1) = 1,$$

the interval

$$(4) \quad I_n = \left[ \frac{4n}{5}, 4n + 5 \right]$$

contains a good integer. Since by the lemma (3) holds for infinitely many positive integers  $n$ , the desired result follows.

To prove our assertion we fix a positive integer  $n$ , for which (3) holds. We may suppose  $\lambda(n) = -1$ , since otherwise  $n \in I_n$  is good, and we are done. Put  $N = 4n$ , and note that, by construction,  $N$  is divisible by 3, 4 and 5. From (3) we get, using the multiplicativity of the function  $\lambda$ ,

$$\lambda(N \pm 4) = \lambda(4(n \pm 1)) = \lambda(4) \lambda(n \pm 1) = 1,$$

and our assumption  $\lambda(n) = -1$  implies

$$\lambda(N) = \lambda(4n) = \lambda(4) \lambda(n) = -1.$$

If now

$$\lambda(N+5) = \lambda(N-5) = -1,$$