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has infinitely many solutions, provided it has at least one solution. It would be interesting to have an analogous result for three (or more) consecutive values, but the above method does not work in this case.

3. Proof of the Theorem, beginning

We shall show here that each of the equations

$$\lambda(n) = \lambda(n+1) = \lambda(n-1) = 1 \tag{2}$$

and

$$\lambda(n) = \lambda(n+1) = \lambda(n-1) = -1 \tag{2'}$$

has infinitely many solutions. Since the arguments for the two cases are completely symmetric, we shall carry out the proof only in the case of equation (2).

Call an integer $n \geqslant 2$ "good", if (2) holds for this n. We have to show that there are infinitely many good integers. To this end we shall show that for any positive integer n satisfying

$$n \equiv 0 \ (\text{mod } 15), \quad \lambda(n+1) = \lambda(n-1) = 1, \tag{3}$$

the interval

$$I_n = \left[\frac{4n}{5}, 4n + 5\right] \tag{4}$$

contains a good integer. Since by the lemma (3) holds for infinitely many positive integers n, the desired result follows.

To prove our assertion we fix a positive integer n, for which (3) holds. We may suppose $\lambda(n) = -1$, since otherwise $n \in I_n$ is good, and we are done. Put $N = 4n$, and note that, by construction, N is divisible by 3, 4 and 5. From (3) we get, using the multiplicativity of the function λ,

$$\lambda(N \pm 4) = \lambda(4(n \pm 1)) = \lambda(4)\,\lambda(n \pm 1) = 1,$$

and our assumption $\lambda(n) = -1$ implies

$$\lambda(N) = \lambda(4n) = \lambda(4)\,\lambda(n) = -1.$$

If now

$$\lambda(N+5) = \lambda(N-5) = -1,$$

then

$$\lambda\left(\frac{N}{5} \pm 1\right) = \frac{\lambda(N \pm 5)}{\lambda(5)} = 1 = -\lambda(N) = \lambda\left(\frac{N}{5}\right),$$

and $N/5 = 4n/5 \in I_n$ is good. We may therefore suppose that at least one of values $\lambda(N+5)$ and $\lambda(N-5)$ equals 1.

For definiteness we shall assume $\lambda(N+5) = 1$; the other case is treated in exactly the same way.

If $\lambda(N+3) = 1$ or $\lambda(N+6) = 1$, then $N + 4 \in I_n$ or $N + 5 \in I_n$ is good. But in the remaining case

$$\lambda(N+3) = \lambda(N+6) = -1$$

we have

$$\lambda\left(\frac{N}{3}\right) = \lambda\left(\frac{N}{3} + 1\right) = \lambda\left(\frac{N}{3} + 2\right) = 1,$$

so that $(N+3)/3 \in I_n$ is good.

Thus (3) implies the existence of a good integer in the interval (4), as we had to show.

4. Proof of the Theorem, conclusion

So far we have proved that (1) has infinitely many solutions in the cases $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$. But this obviously implies that for each of the triples $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, -1), (-1, -1, 1), (1, -1, -1)$ and $(-1, 1, 1)$ there are also infinitely many solutions to (1). It remains therefore to consider the triples $(1, -1, 1)$ and $(-1, 1, -1)$. Since the arguments in both cases are the same (with $+1$ and -1 interchanged), we shall confine ourselves to the case $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$. Accordingly, we call $n \geqslant 2$ good, whenever

$$\lambda(n+1) = \lambda(n-1) = 1, \quad \lambda(n) = -1.$$

We have to show that there are infinitely many such n.

Suppose, to get a contradiction, that there are only finitely many good integers, all of them $\leqslant N_0$, say. Suppose further that

$$\lambda(n) = 1 (m_0 \leqslant n \leqslant n_0) \tag{5}$$

holds for some integers $n_0 > m_0 \geqslant 2N_0$. We shall show that then