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0.5. In the situation of § 0.2, $u(x) = c_0(x)f(x)^s + \dots$ satisfies $\text{supp } \text{sp}(u(x)) = \{\pm df(x)\}$. Therefore $P_m(x, df)$ must be zero if $P(x, \partial)u(x) = 0$. In fact otherwise the bijectivity of $P: \mathcal{C}_M \rightarrow \mathcal{C}_M$ implies $\text{sp}(u) = 0$.

0.6. Such a method of studying functions or differential equations locally on the cotangent bundle is called microlocal analysis. After Sato's discovery of microfunctions, microlocal analysis was studied intensively in Sato-Kawai-Kashiwara [SKK].

Also L. Hörmander [H] worked in the C^∞ -case. Since then, microlocal analysis has been one of the most fundamental tools in the theory of linear partial differential equations.

§ 1. SYSTEMS OF DIFFERENTIAL EQUATIONS (See [O], [Bj])

1.1. Let X be a complex manifold. A system of linear differential equations can be written in the form

$$(1.1.1) \quad \sum_{j=1}^{N_0} P_{ij}(x, \partial)u_j = 0, \quad i = 1, 2, \dots, N_1.$$

Here u_1, \dots, u_{N_0} denote unknown functions and the $P_{ij}(x, \partial)$ are differential operators on X . The holomorphic function solutions of (1.1.1) are simply the kernel of the homomorphism

$$(1.1.2) \quad P: \mathcal{O}_X^{N_0} \rightarrow \mathcal{O}_X^{N_1}$$

which assigns (v_1, \dots, v_{N_1}) to (u_1, \dots, u_{N_0}) , where $v_i = \sum_j P_{ij}(x, \partial)u_j$.

Let us denote by \mathcal{D}_X the ring of differential operators with holomorphic coefficients. Then

$$(1.1.3) \quad P: \mathcal{D}_X^{N_1} \rightarrow \mathcal{D}_X^{N_0}$$

given by (Q_1, \dots, Q_{N_1}) to $(\sum Q_i P_{i1}, \dots, \sum Q_i P_{iN_0})$ is a left \mathcal{D}_X -linear homomorphism. If we denote by \mathcal{M} the cokernel of (1.1.3), then \mathcal{M} becomes a left \mathcal{D}_X -module and $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is the kernel of (1.1.2). This means that the set of holomorphic solutions to $Pu = 0$ depends only on \mathcal{M} .

For this reason we mean by a system of linear differential equations a left \mathcal{D}_X -module.

1.2. Let us take a local coordinate system (x_1, \dots, x_n) of X . Then any differential operator P can be written in the form

$$(1.2.1) \quad P(x, \partial) = \sum_{\alpha \in \mathbf{N}^n} a_\alpha(x) \partial^\alpha$$

where $\partial^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and the $a_\alpha(x)$ are holomorphic functions. For $j \in \mathbf{N}$, we set

$$P_j(x, \xi) = \sum_{|\alpha|=j} a_\alpha(x) \xi^\alpha,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$, and we call $\{P_j(x, \xi)\}$ the *total symbol* of P . The largest m such that $P_m \neq 0$ is called the *order* of P and P_m is called the *principal symbol* of P and denoted by $\sigma(P)$.

Let us denote by T^*X the cotangent bundle of X , and let

$$(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$$

be the associated coordinates of T^*X . It is a classical result that if we consider $\sigma(P)$ as a function on T^*X , then this does not depend on our choice of the local coordinate system (x_1, \dots, x_n) .

1.3. Let M be a real analytic manifold, and X its complexification, e.g., $M = \mathbf{R}^n \subset X = \mathbf{C}^n$. Let P be a differential operator on X . When $\sigma(P)(x, \xi) \neq 0$ for $(x, \xi) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$, P is called an elliptic differential operator. In this case, we have the following result.

PROPOSITION 1.3.1. *If u is a hyperfunction (or distribution) on M and Pu is real analytic, then u is real analytic. More precisely if we denote by \mathcal{A} the sheaf of real analytic functions on M and by \mathcal{B} (resp. $\mathcal{D}b$) the sheaf of hyperfunctions (resp. distributions) on M , then $P: \mathcal{B}/\mathcal{A} \rightarrow \mathcal{B}/\mathcal{A}$ (resp. $P: \mathcal{D}b/\mathcal{A} \rightarrow \mathcal{D}b/\mathcal{A}$) is a sheaf isomorphism.*

This suggests that if $\sigma(P)(x, \xi) \neq 0$, we can consider the inverse P^{-1} in a certain sense. Since (x, ξ) is a point of the cotangent bundle, P^{-1} is attached to the cotangent bundle.

In fact, as we shall see in the sequel, we can construct a sheaf of rings \mathcal{E}_X on T^*X such that $\mathcal{D}_X \subset \pi_* \mathcal{E}_X$, where π is the canonical projection $T^*X \rightarrow X$. Moreover if $P \in \mathcal{D}_X$ satisfies $\sigma(P)(x, \xi) \neq 0$ at a point $(x, \xi) \in T^*X$, then P^{-1} exists as a section of \mathcal{E}_X on a neighborhood of (x, ξ) .

This can be compared to the analogous phenomena for polynomial rings, as shown in the following table.