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$\mathbb{C}[x_1, \dots, x_n]$	\mathcal{D}_X
\mathbb{C}^n	T^*X
the sheaf $\mathcal{O}_{\mathbb{C}^n}$ of holomorphic functions	\mathcal{E}_X

§ 2. MICRO-DIFFERENTIAL OPERATORS (See [SKK], [Bj], [S], [K2])

2.1. Let X be an n -dimensional complex manifold and let $\pi_X: T^*X \rightarrow X$ be the cotangent bundle of X . Let us take a local coordinate system (x_1, \dots, x_n) of X and the associated coordinates $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$ of T^*X . For a differential operator P , let $\{P_j(x, \xi)\}$ be the total symbol of P as in § 1.2. We sometimes write $P = \Sigma P_j(x, \partial)$.

Let $Q = \Sigma Q_j(x, \partial)$ be another differential operator. Set $S = P + Q$ and $R = PQ$. Then the total symbols $\{S_j\}$ and $\{R_j\}$ of R and S are given explicitly by

$$(2.1.1) \quad S_j = P_j + Q_j$$

$$(2.1.2) \quad R_l = \sum_{\substack{l=j+k-|\alpha| \\ \alpha \in \mathbb{N}^n}} \frac{1}{\alpha!} (\partial_{\xi}^{\alpha} P_j) (\partial_x^{\alpha} Q_k)$$

where $\partial_{\xi}^{\alpha} = (\partial/\partial \xi_1)^{\alpha_1} \dots (\partial/\partial \xi_n)^{\alpha_n}$ and $\partial_x^{\alpha} = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$.

The total symbol $\{P_j(x, \xi)\}$ of a differential operator behaves as follows under coordinate transformations. Let (x_1, \dots, x_n) and $(\tilde{x}_1, \dots, \tilde{x}_n)$ be two local coordinate systems. Let (ξ_1, \dots, ξ_n) and $(\tilde{\xi}_1, \dots, \tilde{\xi}_n)$ be related by

$$\xi_k = \sum_j \tilde{\xi}_j \cdot \frac{\partial \tilde{x}_j}{\partial x_k}$$

i.e. $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$ and $(\tilde{x}_1, \dots, \tilde{x}_n; \tilde{\xi}_1, \dots, \tilde{\xi}_n)$ are the associated local coordinate systems of the cotangent bundle T^*X . Let P be a differential operator on X and let $\{P_j(x, \xi)\}$ and $\{\tilde{P}_j(\tilde{x}, \tilde{\xi})\}$ be the total symbols of P with respect to the local coordinate systems (x_1, \dots, x_n) and $(\tilde{x}_1, \dots, \tilde{x}_n)$, respectively. Then one has

$$(2.1.3) \quad \tilde{P}_l(\tilde{x}, \tilde{\xi}) = \sum_{\nu, \alpha_1, \dots, \alpha_\nu} \frac{1}{\nu! \alpha_1! \dots \alpha_\nu!} \langle \tilde{\xi}, \partial_x^{\alpha_1} \tilde{x} \rangle \dots \langle \tilde{\xi}, \partial_x^{\alpha_\nu} \tilde{x} \rangle \partial_{\xi}^{\alpha_1 + \dots + \alpha_\nu} P_j(x, \xi).$$

Here the indices run over $j \in \mathbf{Z}$, $v \in \mathbf{N}$, $\alpha_1, \dots, \alpha_v \in \mathbf{N}^n$ such that $|\alpha_1|, \dots, |\alpha_v| \geq 2$ and $l = j + v - |\alpha_1| - \dots - |\alpha_v|$. For $\beta \in \mathbf{N}^n$, $\langle \tilde{\xi}, \partial_x^\beta \tilde{x} \rangle$ denotes $\sum_j \tilde{\xi}_j \partial^\beta \tilde{x}_j$.

2.2. The total symbol $\{P_j(x, \xi)\}$ of a differential operator is a polynomial in ξ . We shall define microdifferential operators by admitting P_j to be holomorphic in ξ .

For $\lambda \in \mathbf{C}$, let $\mathcal{O}_{T^*X}(\lambda)$ be the sheaf of homogeneous holomorphic functions of degree λ on T^*X , i.e., holomorphic functions $f(x, \xi)$ satisfying

$$(\sum \xi_j \partial / \partial \xi_j - \lambda) f(x, \xi) = 0.$$

Definition 2.2.1. For $\lambda \in \mathbf{C}$ we define the sheaf $\mathcal{E}_X(\lambda)$ on T^*X by

$$\Omega \mapsto \{(P_{\lambda-j}(x, \xi))_{j \in \mathbf{N}}; P_{\lambda-j} \in \Gamma(\Omega; \mathcal{O}_{T^*X}(\lambda-j))\}$$

and satisfies the following conditions (2.2.1)

(2.2.1) for any compact subset K of Ω , there exists a $C_K > 0$ such that

$$\sup_K |P_{\lambda-j}| \leq C_K^{-j} (j!) \quad \text{for all } j > 0.$$

Remark. The growth condition (2.2.1) can be explained as follows. For a differential operator $P = \sum P_j(x, \partial)$, we have

$$P(x, \partial) (\langle x, \xi \rangle + p)^\mu = \sum P_j(x, \xi) \frac{\Gamma(\mu)}{\Gamma(\mu-j+1)} (\langle x, \xi \rangle + p)^{\mu-j}.$$

For $P = (P_{\lambda-j}(x, \xi)) \in \mathcal{E}(\lambda)$ we set, by analogy

$$P(\langle x, \xi \rangle + p)^\mu = \sum_j P_{\lambda-j}(x, \xi) \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda+j+1)} (\langle x, \xi \rangle + p)^{\mu-\lambda+j}.$$

Then the growth condition (2.2.1) is simply the condition that the right hand side converges when $0 < |\langle x, \xi \rangle + p| \ll 1$.

Now, we have the following

PROPOSITION 2.2.2 ([SKK], Chap. II, § 1, [Bj] Chap. IV, § 1).

- (0) $\mathcal{E}_X(\lambda)$ contains $\mathcal{E}_X(\lambda-m)$ as a subsheaf for $m \in \mathbf{N}$.
- (1) Patching by rule (2.1.3) under coordinate transformations, $\mathcal{E}_X(\lambda)$ becomes a sheaf defined globally on T^*X .

(2) By rule (2.1.1), $\mathcal{E}_X(\lambda)$ is a sheaf of \mathbf{C} -vector space on T^*X .

(3) By rule (2.1.2), we can define the "product" homomorphism:

$$\mathcal{E}_X(\lambda) \otimes_{\mathbf{C}} \mathcal{E}_X(\mu) \rightarrow \mathcal{E}_X(\lambda + \mu),$$

which satisfies the associative law.

(4) In particular, $\mathcal{E}_X(0)$ and $\mathcal{E}_X = \bigcup_{m \in \mathbf{Z}} \mathcal{E}_X(m)$ become sheaves of (non commutative) rings on T^*X , with a unit.

The unit is given by $(P_j(x, \xi))$ with $P_j = 1$ for $j = 0$ and $P_j = 0$ for $j \neq 0$.

We define the homomorphism

$$\sigma_\lambda: \mathcal{E}_X(\lambda) \rightarrow \mathcal{O}_{T^*X}(\lambda)$$

by

$$(P_{\lambda-j}) \mapsto P_\lambda.$$

Then, σ_λ is a well-defined homomorphism on T^*X (i.e. compatible with coordinate transformation) and we have an exact sequence

$$0 \rightarrow \mathcal{E}_X(\lambda-1) \rightarrow \mathcal{E}_X(\lambda) \xrightarrow{\sigma_\lambda} \mathcal{O}_{T^*X}(\lambda) \rightarrow 0.$$

Now we have the following proposition, which says that the ring \mathcal{E}_X is a kind of localization of \mathcal{D}_X .

PROPOSITION 2.2.3.

- (1) For $P \in \mathcal{E}(\lambda)$ and $Q \in \mathcal{E}(\mu)$, we have $\sigma_{\lambda+\mu}(PQ) = \sigma_\lambda(P)\sigma_\mu(Q)$.
- (2) ([SKK] Chap. II, Thm. 2.1.1) If $P \in \mathcal{E}(\lambda)$ satisfies $\sigma_\lambda(P)(q) \neq 0$ at $q \in T^*X$, then there exists $Q \in \mathcal{E}(-\lambda)$ such that $PQ = QP = 1$.

The relations between \mathcal{E}_X and \mathcal{D}_X are summarized in the following theorem.

THEOREM 2.2.4 ([SKK], Chap. II, § 3).

- (i) \mathcal{E}_X contains $\pi^{-1}\mathcal{D}_X$ as a subring and is flat over $\pi^{-1}\mathcal{D}_X$.
- (ii) $\mathcal{E}_X|_{T^*_X X} \simeq \mathcal{D}_X$, where $T^*_X X$ is the zero section of T^*X .
- (iii) For a coherent \mathcal{D}_X -module \mathcal{M} , the characteristic variety of \mathcal{M} coincides with the support of $\mathcal{E}_X \otimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathcal{M}$.