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§ 3. THE ALGEBRAIC PROPERTIES OF \mathcal{E} (See [SKK], [Bj])

3.1. In the preceding section, we introduced the notion of micro-differential operators. The ring \mathcal{E} of micro-differential operators has nice algebraic properties similar to those of the ring of holomorphic functions.

Let us recall some definitions of finiteness properties.

Definition 3.1.1. Let \mathcal{A} be a sheaf of rings on a topological space S .

- (1) An \mathcal{A} -module \mathcal{M} is called *of finite type* (resp. *of finite presentation*) if for any point $x \in X$ there exists a neighborhood U and an exact sequence $0 \leftarrow \mathcal{M}|_U \leftarrow \mathcal{A}^p|_U$ (resp. $0 \leftarrow \mathcal{M}|_U \leftarrow \mathcal{A}^p|_U \leftarrow \mathcal{A}^q|_U$).
- (2) \mathcal{M} is called *pseudo-coherent*, if any submodule of finite type defined on an open subset is of finite presentation. If \mathcal{M} is pseudo-coherent and of finite type, then \mathcal{M} is called *coherent*.
- (3) \mathcal{M} is called *Noetherian* if \mathcal{M} satisfies the following properties:
 - (a) \mathcal{M} is coherent.
 - (b) For any $x \in X$, \mathcal{M}_x is a Noetherian \mathcal{A}_x -module (i.e. any increasing sequence of \mathcal{A}_x -submodules is stationary).
 - (c) For any open subset U , any increasing sequence of coherent $(\mathcal{A}|_U)$ -submodules of $\mathcal{M}|_U$ is locally stationary.

As for the sheaf of holomorphic functions, we have

THEOREM 3.1.1 ([SKK] Chap. II, Thm. 3.4.1, Prop. 3.2.7). *Let \mathring{T}^*X denote the complement of the zero section in T^*X .*

- (1) \mathcal{E}_X and $\mathcal{E}_X(0)$ are Noetherian rings on T^*X .
- (2) \mathcal{E}_X is flat over $\pi^{-1}\mathcal{D}_X$.
- (3) $\mathcal{E}_X(\lambda)|_{\mathring{T}^*X}$ is a Noetherian $\mathcal{E}_X(0)|_{\mathring{T}^*X}$ -module.
- (4) For $p \in T^*X$, $\mathcal{E}_X(0)_p$ is a local ring with the residual field \mathbb{C} .
- (5) A coherent \mathcal{E}_X -module is pseudo-coherent over $\mathcal{E}_X(0)$.

§ 4. VARIANTS OF \mathcal{E} (See [SKK], [Bj], [S])

4.1. We have defined the sheaf of rings \mathcal{E} . However we can introduce other sheaves of rings, similar to \mathcal{E} , which makes the theory transparent.

4.2. The sheaf $\hat{\mathcal{E}} = \varprojlim_{m \in \mathbf{N}} \mathcal{E}/\mathcal{E}(-m)$ is called the sheaf of *formal micro-differential operators*. This is nothing but the sheaf similar to \mathcal{E} , obtained by dropping the growth condition (2.2.1).

4.3. We can define the sheaf \mathcal{E}^∞ of micro-differential operators of infinite order ([SKK]). For an open $\Omega \subset \mathbf{C}^n$, we set

$$\Gamma(\Omega; \mathcal{E}^\infty) = \{(p_j)_{j \in \mathbf{Z}}; p_j \in \Gamma(\Omega; \mathcal{O}_{T^*X}(j))\}$$

satisfying the following conditions (4.3.1) and (4.3.2).

(4.3.1) For any compact set $K \subset \Omega$, there is a $C_K > 0$ such that $\sup_K |p_j| \leq C_K^{-j} (-j)!$ for $j < 0$.

(4.3.2) For any compact set $K \subset \Omega$ and any $\varepsilon > 0$, there exists a $C_{K, \varepsilon} > 0$ such that

$$\sup_K |p_j| \leq C_{K, \varepsilon} \frac{\varepsilon^j}{j!} \quad \text{for } j \geq 1.$$

4.4. We can also define the sheaf $\mathcal{E}^{\mathbf{R}}$ on T^*X by $\mathcal{H}^n(\mu_\Delta(\mathcal{O}_{X \times X}^{(0, n)}))$. (See [KS] Chap. II, [SKK]). Here $n = \dim X$, $\mathcal{O}_{X \times X}^{(0, n)}$ is the sheaf of holomorphic forms on $X \times X$ which are n -forms with respect to the second variable, and μ_Δ is the micro-localization with respect to the diagonal set of $X \times X$ (See [SKK] Chap. II for the details).

4.5. We have $\mathcal{E}_X \subset \mathcal{E}_X^\infty \subset \mathcal{E}_X^{\mathbf{R}}$, $\mathcal{E}_X \subset \hat{\mathcal{E}}_X$. Moreover, \mathcal{E}_X^∞ , $\mathcal{E}_X^{\mathbf{R}}$ and $\hat{\mathcal{E}}_X$ are faithfully flat over \mathcal{E}_X . The sheaf $\hat{\mathcal{E}}_X$ is Noetherian. The sheaf $\mathcal{E}_X^{\mathbf{R}}$ contains $\mathcal{E}_X(\lambda)$'s compatible with the multiplication.

4.6. If we denote by γ the projection map $T^*X \rightarrow T^*X/\mathbf{C}^*$, then $R^j \gamma_* \mathcal{E}^{\mathbf{R}} = 0$ for $j \neq 0$ and $\mathcal{E}^\infty = \gamma^{-1} \gamma_* \mathcal{E}^{\mathbf{R}}$.

4.7. In [SKK], \mathcal{E} , $\hat{\mathcal{E}}$, and \mathcal{E}^∞ are denoted by \mathcal{P}^f , $\hat{\mathcal{P}}$ and \mathcal{P} .

4.8. To explain the differences between \mathcal{E} , \mathcal{E}^∞ , $\mathcal{E}^{\mathbf{R}}$ and $\hat{\mathcal{E}}$, we shall take the following example. Let X be a complex manifold and Y a hypersurface of X . We shall take local coordinates (x_1, \dots, x_n) of X such that Y is given by $x_1 = 0$. The \mathcal{D}_X -module $\mathcal{D}_X/\mathcal{D}_X x_1 + \sum_{j>1} \mathcal{D}_X \partial_j$ is denoted by $\mathcal{B}_{Y|X}$. Set

$$\begin{aligned} \mathcal{C}_{Y|X} &= \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{B}_{Y|X}, \quad \widehat{\mathcal{C}}_{Y|X} = \widehat{\mathcal{E}}_X \otimes_{\mathcal{E}_X} \mathcal{C}_{Y|X}, \\ \mathcal{C}_{Y|X}^\infty &= \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{C}_{Y|X} \quad \text{and} \quad \mathcal{C}_{Y|X}^{\mathbf{R}} = \mathcal{E}_X^{\mathbf{R}} \otimes_{\mathcal{E}_X} \mathcal{C}_{Y|X}. \end{aligned}$$

Then we have, setting $p = (0, dx_1)$, $x_0 = 0$

$$\begin{aligned} \mathcal{C}_{Y|X,p} &= \{a + b \log x_1; a \in \mathcal{O}_{X,x_0}[1/x_1], b \in \mathcal{O}_{X,x_0}\} / \mathcal{O}_{X,x_0} \\ &\cong (\mathcal{O}_{X,x_0}[1/x_1] / \mathcal{O}_{X,x_0}) \oplus \mathcal{O}_{X,x_0} \\ \widehat{\mathcal{C}}_{Y|X,p} &= \{a + b \log x_1; a \in \mathcal{O}_{X,x_0}[1/x_1], b \in \widehat{\mathcal{O}}_{X|Y,x_0}\} / \mathcal{O}_{X,x_0} \\ &\cong (\mathcal{O}_{X,x_0}[1/x_1] / \mathcal{O}_{X,x_0}) \oplus \widehat{\mathcal{O}}_{X|Y,x_0}. \end{aligned}$$

Here $\widehat{\mathcal{O}}_{X|Y} = \varprojlim \mathcal{O}_X / x_1^m \mathcal{O}_X$ is the sheaf of formal power series in the x_1 -direction.

$$\mathcal{C}_{Y|X,p}^\infty = \{a + b \log x_1; a \in (j_* j^{-1} \mathcal{O}_X)_{x_0}, b \in \mathcal{O}_{X,x_0}\} / \mathcal{O}_{X,x_0}$$

where j is the open embedding $X \setminus Y \hookrightarrow X$.

$$\mathcal{C}_{Y|X,p}^{\mathbf{R}} = \varinjlim_U \mathcal{O}(U) / \mathcal{O}_{X,x_0}.$$

Here U ranges over the set of open subsets of the form

$$\{x \in X; |x| < \varepsilon, \operatorname{Re} x_1 < \varepsilon \operatorname{Im} x_1\}.$$

4.8. If we use \mathcal{E}_X^∞ , the structure of \mathcal{E} -modules becomes simpler. We just mention two theorems in this direction.

THEOREM 4.8.1 ([KK] Thm. 5.2.1). *Let \mathcal{M} be a holonomic \mathcal{E}_X -module. Then there exists a (unique) regular holonomic \mathcal{E}_X -module \mathcal{M}_{reg} such that*

$$\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M} \cong \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}_{\text{reg}}.$$

THEOREM 4.8.2 ([SKK] Chap. II, Thm. 5.3.1). *Let X and Y be complex manifolds and let T_Y^*Y be the zero section of T^*Y . If \mathcal{M} is an $\mathcal{E}_{X \times Y}$ -module whose support is contained in $T^*X \times T_Y^*Y$, then there exists a (locally) coherent \mathcal{E}_X -module \mathcal{L} such that*

$$\mathcal{E}_{X \times Y}^\infty \otimes_{\mathcal{E}_{X \times Y}} \mathcal{M} \cong \mathcal{E}_{X \times Y}^\infty \otimes_{\mathcal{E}_{X \times Y}} (\mathcal{L} \widehat{\otimes} \mathcal{O}_Y).$$

Here $\widehat{\otimes}$ denotes the exterior tensor product. (See § 8).