

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 32 (1986)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INTRODUCTION TO MICROLOCAL ANALYSIS
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Kapitel: §10. Structure of Regular Holonomic E-Modules (See [SKK], [KK])
DOI: <https://doi.org/10.5169/seals-55089>

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9.4. Noting that any nowhere dense closed analytic subset of a Lagrangean variety is never involutive, Theorem 9.2.3 implies the following theorem.

THEOREM 9.4.1. *Let \mathcal{M} be a holonomic \mathcal{E}_X -module. Then the following conditions are equivalent.*

- (i) *There exists a Lagrangean subvariety Λ such that \mathcal{M} has regular singularities along Λ .*
- (ii) *For any involutive subvariety Λ which contains $\text{Supp } \mathcal{M}$, \mathcal{M} has regular singularities along Λ .*
- (iii) *There exists an open dense subset Ω of $\text{Supp } \mathcal{M}$ such that \mathcal{M} has regular singularities along $\text{Supp } \mathcal{M}$ on Ω .*

If these equivalent conditions are satisfied, we say that \mathcal{M} is a *regular holonomic \mathcal{E}_X -module*.

The following properties are almost immediate.

THEOREM 9.4.2.

- (i) *Let $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$ be an exact sequence of three coherent \mathcal{E}_X -modules. If two of them are regular holonomic then so is the third.*
- (ii) *If \mathcal{M} is regular holonomic, its dual \mathcal{M}^* is also regular holonomic.*

We just mention another analytic property of regular holonomic modules, which generalizes the fact that a formal solution of an ordinary differential equation with regular singularity converges.

THEOREM 9.4.3 ([KK] Theorem 6.1.3). *If \mathcal{M} and \mathcal{N} are regular holonomic \mathcal{E}_X -modules, then $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \widehat{\mathcal{E}}_X \otimes_{\mathcal{E}_X} \mathcal{N})$ and $\mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{N})$ are isomorphisms.*

§ 10. STRUCTURE OF REGULAR HOLONOMIC \mathcal{E} -MODULES

(See [SKK], [KK])

10.1. Let Λ be a Lagrangean submanifold of T^*X . We define \mathcal{I}_Λ and \mathcal{E}_Λ as in § 9.2.

Then $\mathcal{E}_\Lambda(-1) = \mathcal{E}_\Lambda \cdot \mathcal{E}(-1)$ is a two-sided ideal of \mathcal{E}_Λ and $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$ is a sheaf of rings which contains $\mathcal{O}_\Lambda(0) = \mathcal{E}(0)/\mathcal{I}_\Lambda(-1)$, the sheaf of homogeneous functions on Λ .

Let us take an invertible \mathcal{O}_Λ -module \mathcal{L} such that $\mathcal{L}^{\otimes 2} \cong \omega_\Lambda \otimes_{\theta_X} \omega_X^{\otimes -1}$. Such an \mathcal{L} exists at least locally. For $P = P_1(x, \partial) + P_0(x, \partial) + \dots \in \mathcal{I}$ we define, for $\varphi \in \mathcal{O}_\Lambda$ and an invertible section s of \mathcal{L} ,

$$L(P)(\varphi s) = \left\{ H_{P_1}(\varphi) + \frac{1}{2} \varphi \frac{L_{H_{P_1}}(s^{\otimes 2} \otimes dx)}{s^{\otimes 2} \otimes dx} + \left(P_0 - \frac{1}{2} \sum_{j=1}^n \frac{\partial^2 P_1}{\partial x_i \partial \xi_i} \right) \varphi \right\} s.$$

Here $dx = dx_1 \wedge \dots \wedge dx_n \in \omega_X$ and $s^{\otimes 2} \otimes dx$ is regarded as a section of ω_Λ . The Lie derivative $L_{H_{P_1}}$ of H_{P_1} operates on ω_Λ as the first order differential operators so that $L_{H_{P_1}}(s^{\otimes 2} \otimes dx)$ is a section of ω_Λ and $L_{H_{P_1}}(s^{\otimes 2} \otimes dx)/s^{\otimes 2} \otimes dx$ is a function on Λ .

We thus obtain $L: \mathcal{I}_\Lambda \rightarrow \mathcal{E}nd_{\mathcal{C}}(\mathcal{L})$. Then this does not depend on the choice of local coordinate system and moreover it extends to the ring homomorphism $L: \mathcal{E}_\Lambda \rightarrow \mathcal{E}nd_{\mathcal{C}}(\mathcal{L})$. Since the image is contained in the differential endomorphism of \mathcal{L} , we obtain the ring homomorphism $L: \mathcal{E}_\Lambda \rightarrow \mathcal{L} \otimes_{\mathcal{O}_\Lambda} \mathcal{D}_\Lambda \otimes_{\mathcal{O}_\Lambda} \mathcal{L}^{\otimes -1}$.

PROPOSITION 10.1.1. *By $L, \mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$ coincides with the subsheaf of $\mathcal{L} \otimes_{\mathcal{O}_\Lambda} \mathcal{D}_\Lambda \otimes_{\mathcal{O}_\Lambda} \mathcal{L}^{\otimes -1}$ consisting of differential endomorphisms of \mathcal{L} homogeneous of degree 0.*

If we take

$$\mathcal{I}_\Lambda \in \mathfrak{D} = \mathfrak{D}_1(x, \partial) + \mathfrak{D}_0(x, \partial) + \dots$$

such that $d\mathfrak{D}_1 \equiv -\theta_X \text{ mod } I_\Lambda \Omega^1$ and

$$\frac{1}{2} \sum \frac{\partial^2 \mathfrak{D}_1}{\partial x_i \partial \xi_i} \equiv \mathfrak{D}_0(x, \xi) \text{ mod } \mathcal{I}_\Lambda$$

then $L(\mathfrak{D})$ gives the Euler operator of \mathcal{L} . Such a \mathfrak{D} is unique modulo $\mathcal{I}_\Lambda^2(-1) = \mathcal{E}_\Lambda(-1) \cap \mathcal{E}_X(1)$.

10.2. Let \mathcal{M} be a regular holonomic \mathcal{E}_X -module whose support is Λ . Let \mathcal{M}_0 be a coherent sub- \mathcal{E}_Λ -module of \mathcal{M} which generates \mathcal{M} . Such an \mathcal{M}_0 is called a *saturated lattice* of \mathcal{M} . Then $\bar{\mathcal{M}} = \mathcal{M}_0/\mathcal{E}(-1)\mathcal{M}_0$ is an $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$ -module, which is coherent over $\mathcal{O}_\Lambda(0)$.

Since a coherent sheaf with integrable connection is locally free, we have

LEMMA 10.2.1. *$\bar{\mathcal{M}}$ is a locally free $\mathcal{O}_\Lambda(0)$ -module of finite rank.*

Since \mathfrak{g} belongs to the center of $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$, \mathfrak{g} can be considered as an endomorphism of $\mathcal{H}om_{\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)}(\bar{\mathcal{M}}, \mathcal{L})$, which is a locally constant sheaf on Λ . Its eigenvalues are called the *order* of \mathcal{M} with respect to \mathcal{M}_0 .

10.3. Let us take a section $G \subset \mathbf{C}$ of $\mathbf{C} \rightarrow \mathbf{C}/\mathbf{Z}$. Then there exists a unique saturated lattice \mathcal{M}_0 such that the orders of \mathcal{M} with respect to \mathcal{M}_0 are contained in G (See [K4]). Then

$$\mathcal{F} = \mathcal{H}om_{\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)}(\bar{\mathcal{M}}, \mathcal{L})$$

and

$$M = \exp 2\pi i \mathfrak{g} \in \mathcal{A}ut(\mathcal{F})$$

does not depend on the choice of G .

THEOREM 10.3.1 ([KK] Chapter I, § 3). *Assume that there exists an invertible \mathcal{O}_Λ -module \mathcal{L} such that $\mathcal{L}^{\otimes 2} = \omega_\Lambda \otimes \omega_X^{\otimes -1}$. Then the category of regular holonomic \mathcal{E}_X -modules with support in Λ is equivalent to the category of (\mathcal{F}, M) 's where \mathcal{F} is a locally constant \mathbf{C}_Λ -module and $M \in \mathcal{A}ut_{\mathbf{C}}(\mathcal{F})$.*

10.4. If $u \in \mathcal{M}$, then the solution to $L(P)\varphi = 0$ for $P \in \mathcal{E}_\Lambda$ with $Pu = 0$ is called a principal symbol of u and denoted by $\sigma(u)$. The homogeneous degree of $\sigma(u)$ is called the order of u . In the terminology of § 10.2, the principal symbol is a section of $\mathcal{H}om_{\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)}(\mathcal{E}_\Lambda u/\mathcal{E}_\Lambda(-1)u, \mathcal{L})$ and the order is the eigenvalue of \mathfrak{g} in $\mathcal{H}om_{\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)}(\mathcal{E}_\Lambda u/\mathcal{E}_\Lambda(-1)u, \mathcal{L})$.

10.4. When the characteristic variety is not smooth, we don't know much about the structure of holonomic systems. In this direction, we have

THEOREM 10.4.1 ([K-K] Theorem 1.2.2). *Let Z be a closed analytic subset of an open subset Ω of T^*X , $n = \dim X$, and let \mathcal{M} and \mathcal{N} be holonomic $\mathcal{E}_X|_\Omega$ -modules.*

(i) *If $\dim Z \leq n-1$, then*

$$\Gamma(\Omega; \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N})) \rightarrow \Gamma(\Omega \setminus Z; \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}))$$

is injective.

(ii) *If $\dim Z \leq n-2$, then*

$$\Gamma(\Omega; \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N})) \rightarrow \Gamma(\Omega \setminus Z; \mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{N}))$$

is an isomorphism.

In particular if $\text{Supp } \mathcal{M} \subset \Lambda_1 \cup \Lambda_2$ and if $\dim(\Lambda_1 \cap \Lambda_2) \leq n-2$, then \mathcal{M} is a direct sum of two holonomic \mathcal{E}_X -modules supported on Λ_1 and Λ_2 , respectively.

Here is another type of theorem.

THEOREM 10.4.3 ([SKKO]). Let $\mathcal{M} = \mathcal{E}u = \mathcal{E}/\mathcal{I}$ be a holonomic \mathcal{E} -module defined on a neighborhood of $p \in T^*X$. Assume $\text{Supp } \mathcal{M} = \Lambda_1 \cup \Lambda_2$ and

- (i) Λ_1, Λ_2 and $\Lambda_1 \cap \Lambda_2$ are non-singular and $\dim \Lambda_1 = \dim \Lambda_2 = n, \dim(\Lambda_1 \cap \Lambda_2) = n-1$.
- (ii) $T_{p'} \Lambda_1 \cap T_{p'} \Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$ for any p' in a neighborhood of p in $\Lambda_1 \cap \Lambda_2$.
- (iii) The symbol ideal of \mathcal{I} coincides with the ideal of functions vanishing on $\Lambda_1 \cup \Lambda_2$.

Setting $k = \text{ord}_{\Lambda_1} u - \text{ord}_{\Lambda_2} u - 1/2$, we have

- (a) \mathcal{M} has a non-zero quotient supported on $\Lambda_1 \Leftrightarrow \mathcal{M}$ has a non-zero submodule supported on $\Lambda_2 \Leftrightarrow k \in \mathbf{Z}$.
- (b) \mathcal{M}_p is a simple \mathcal{E}_p -module $\Leftrightarrow k \notin \mathbf{Z}$.

Sketch of the proof. By a quantized contact transformation, we can transform p, Λ_1, Λ_2 and \mathcal{I} as follows:

$$p = (0, dx_1)$$

$$\Lambda_1 = \{(x, \xi); x_1 = \xi_2 = \dots = \xi_n = 0\}$$

$$\Lambda_2 = \{(x, \xi); x_1 = x_2 = \xi_3 = \dots = \xi_n = 0\}$$

$$\mathcal{I} = \mathcal{E}(x_1 \partial_1 - \lambda) + \mathcal{E}(x_2 \partial_2 - \mu) + \sum_{j>2} \mathcal{E} \partial_j$$

In this case, we can easily check the theorem.

§ 11. APPLICATION TO THE b -FUNCTION (see [SKKO])

11.1. As one of the most successful application of microlocal analysis, we shall sketch here how to calculate the b -function of a function under certain conditions.