Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 32 (1986)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TREES, TAIL WAGGING AND GROUP PRESENTATIONS

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Kapitel: 3. Tail wagging

DOI: https://doi.org/10.5169/seals-55090

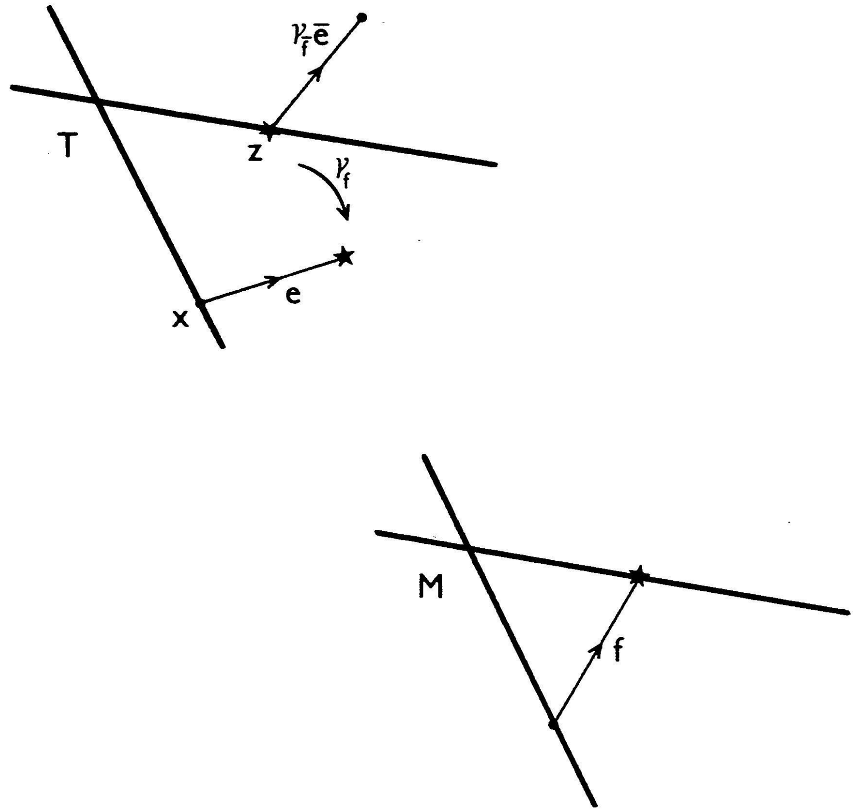

FIGURE 1

3. TAIL WAGGING

With the notation established above let $*G_w$ denote the free product of the stablizers of the vertices of T, and F the free group generated by symbols λ_f, one for each edge f of X/G. Let R be the normal consequence in $(*G_w)*F$ of the words

$$\lambda_f \quad (f \text{ an edge of } M),$$

$$\lambda_{\bar f}\,\lambda_f \quad \text{and}$$

$$\lambda_{\bar f}g_x\lambda_f(\gamma_{\bar f}g\gamma_f)_z^{-1}$$

We shall produce an isomorphism

$$\psi: G \to [(\ast G_w) \ast F]/R\,.$$

Choose a vertex v of T as base point. If $g \in G$ fixes v set

$$\psi(g) = g_v R$$

where as usual g_v is the element g interpreted as a member of G_v. If g moves v then it sends it outside T because no two vertices of T lie in the same orbit. Let $e_1 e_2 \ldots e_n$ be the geodesic which joins v to gv and suppose e_m is the first edge that is *not* in T. The path $e_m e_{m+1} \ldots e_n$ will be called the *tail* of $\overrightarrow{v\,gv}$. Let x_1 be the initial vertex of e_m. Project e_m into X/G to give an edge f_1. The canonical lift e^1 of f_1 into X has its initial vertex in T, so $i(e^1) = x_1$. Choose an element $a_{x_1} \in G_{x_1}$ which sends e^1 to e_m. Let

$$e_k^1 = (\gamma_{\bar{f}_1} a_{x_1}^{-1}) e_k$$

for $m+1 \leqslant k \leqslant n$, and replace $e_1 e_2 \ldots e_n$ by the new path $e_{m+1}^1 e_{m+2}^1 \ldots e_n^1$. We call this process *tail wagging*. Our new path begins at

$$z_1 = t(\gamma_{\bar{f}_1} e^1) = i(e_{m+1}^1)$$

which is a vertex of T and ends at $(\gamma_{\bar{f}_1} a_{x_1}^{-1} g)v$, see Figure 2. We walk along it to the first point x_2 where it quits T and repeat the above

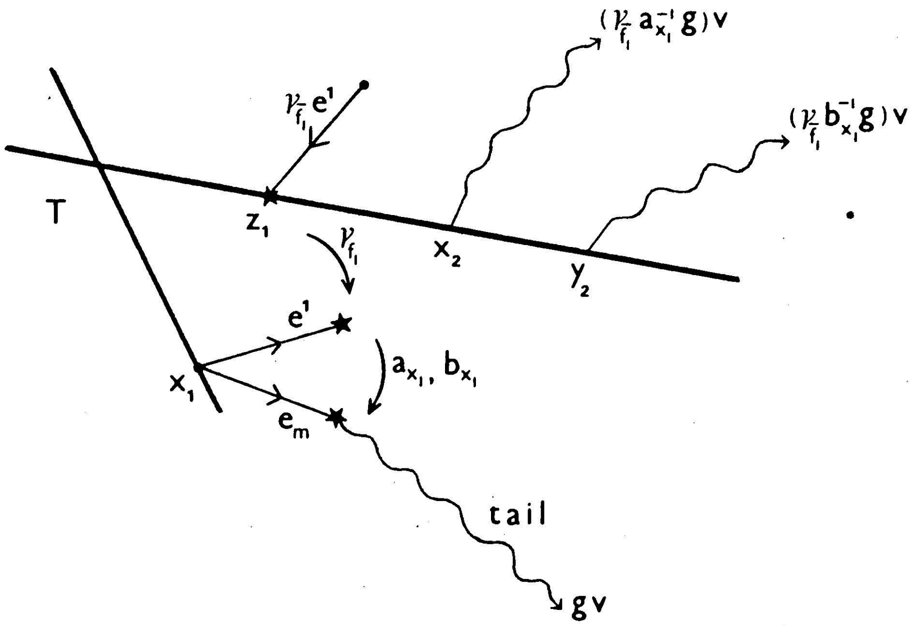

FIGURE 2

procedure. Since we shorten the tail at each step we eventually obtain a path which lies entirely in T and ends at say

$$(\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_2} a_{x_2}^{-1} \gamma_{\bar{f}_1} a_{x_1}^{-1} g)v .$$

Then $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g$ must fix v, say $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g = a_v \in G_v$. We now have

$$g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v$$

and we somewhat optimistically define

$$\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R .$$

4. An inefficient choice

Is ψ well defined? The geodesic from v to gv is certainly unique, as is the first point x_1 where it leaves T and its first edge e_m outside T. Both the edge e^1 and the group element γ_{f_1} are now determined by our original construction. The only ambiguity at this stage is the choice of the element $a_{x_1} \in G_{x_1}$ which maps e^1 to e_m. A different choice b_{x_1} will give a path from z_1 to $(\gamma_{\bar{f}_1} b_{x_1}^{-1} g)v$ which leaves T for the first time at say y_2. The first edge outside T will project to an edge f'_2 of X/G and so on until eventually we have g expressed as

$$g = b_{x_1} \gamma_{f_1} b_{y_2} \gamma_{f'_2} \dots b_{y_s} \gamma_{f'_s} b_v .$$

We must show that $a_{x_1} \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v$ and $b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v$ determine the same left coset of R in $(\ast G_w)\ast F$.

Agree to select a_{x_1} from G_{x_1} so that the tail of the resulting path is as *long* as possible. Continue in this way selecting $a_{x_2}, a_{x_3} \dots$ so as to maximise the length of the tail at each stage. We shall compare any other set of choices with this rather inefficient selection.

Both a_{x_1} and b_{x_1} map e^1 to e_m, so $c = a_{x_1}^{-1} b_{x_1}$ must fix e^1. Also, due to our particular selection of a_{x_1}, the geodesic from z_1 to x_2 is left fixed by $\gamma_{\bar{f}_1} c \, \gamma_{f_1}$. Therefore