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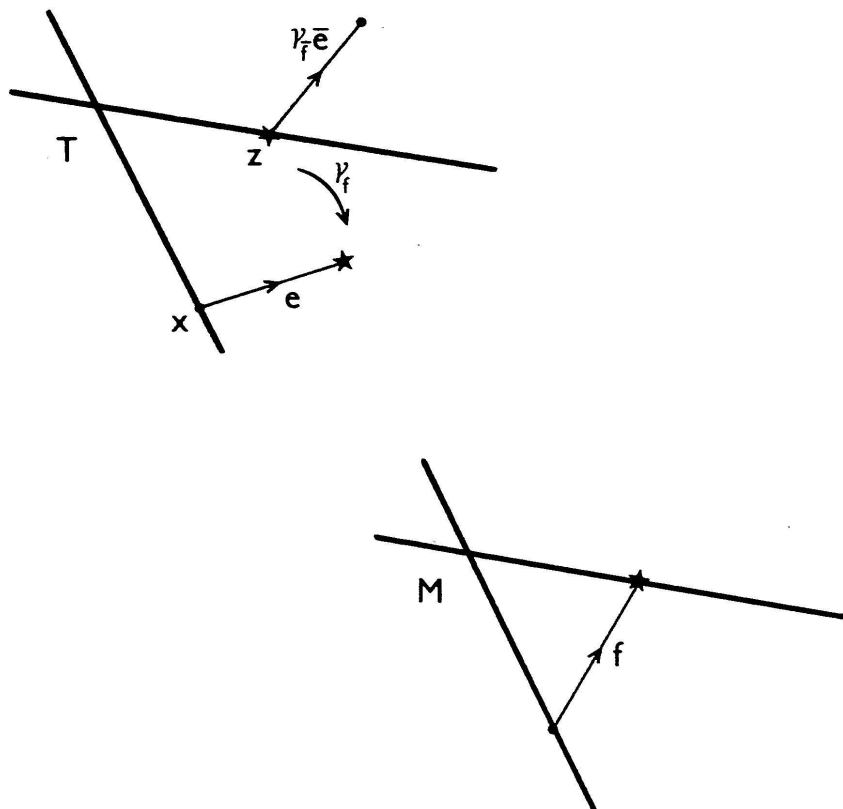


FIGURE 1

### 3. TAIL WAGGING

With the notation established above let  $*G_w$  denote the free product of the stabilizers of the vertices of  $T$ , and  $F$  the free group generated by symbols  $\lambda_f$ , one for each edge  $f$  of  $X/G$ . Let  $R$  be the normal consequence in  $(*G_w)*F$  of the words

$$\begin{aligned} \lambda_f & \quad (f \text{ an edge of } M), \\ \lambda_{\bar{f}} \lambda_f & \quad \text{and} \\ \lambda_{\bar{f}} g_x \lambda_f (\gamma_{\bar{f}} g \gamma_f)_z^{-1} \end{aligned}$$

We shall produce an isomorphism

$$\psi: G \rightarrow [(*G_w)*F]/R.$$

Choose a vertex  $v$  of  $T$  as base point. If  $g \in G$  fixes  $v$  set

$$\psi(g) = g_v R$$

where as usual  $g_v$  is the element  $g$  interpreted as a member of  $G_v$ . If  $g$  moves  $v$  then it sends it outside  $T$  because no two vertices of  $T$  lie in the same orbit. Let  $e_1 e_2 \dots e_n$  be the geodesic which joins  $v$  to  $gv$  and suppose  $e_m$  is the first edge that is *not* in  $T$ . The path  $e_m e_{m+1} \dots e_n$  will be called the *tail* of  $\overrightarrow{v gv}$ . Let  $x_1$  be the initial vertex of  $e_m$ . Project  $e_m$  into  $X/G$  to give an edge  $f_1$ . The canonical lift  $e^1$  of  $f_1$  into  $X$  has its initial vertex in  $T$ , so  $i(e^1) = x_1$ . Choose an element  $a_{x_1} \in G_{x_1}$  which sends  $e^1$  to  $e_m$ . Let

$$e_k^1 = (\gamma_{f_1} a_{x_1}^{-1}) e_k$$

for  $m+1 \leq k \leq n$ , and replace  $e_1 e_2 \dots e_n$  by the new path  $e_{m+1}^1 e_{m+2}^1 \dots e_n^1$ . We call this process *tail wagging*. Our new path begins at

$$z_1 = t(\gamma_{f_1} e^1) = i(e_{m+1}^1)$$

which is a vertex of  $T$  and ends at  $(\gamma_{f_1} a_{x_1}^{-1} g)v$ , see Figure 2. We walk along it to the first point  $x_2$  where it quits  $T$  and repeat the above

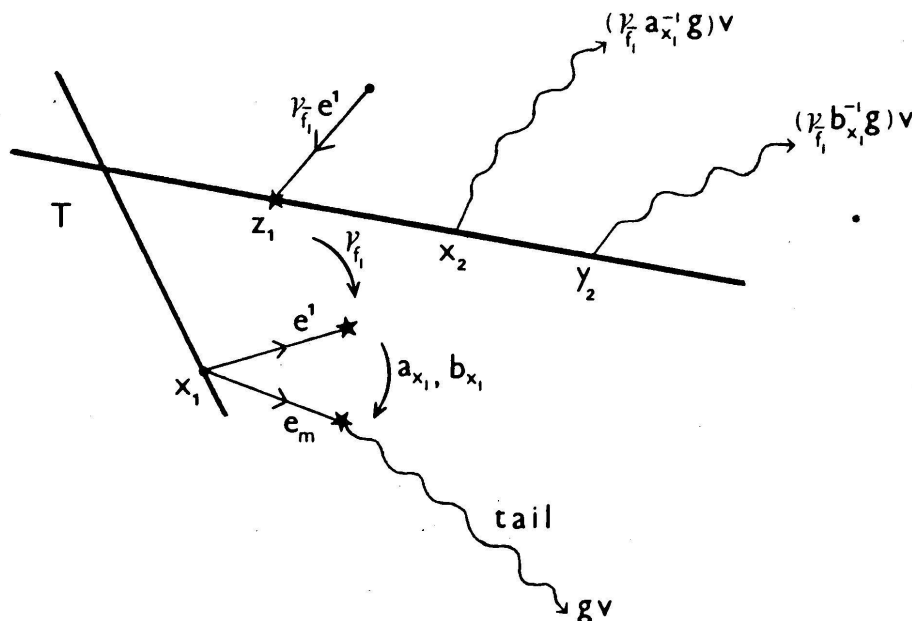


FIGURE 2

procedure. Since we shorten the tail at each step we eventually obtain a path which lies entirely in  $T$  and ends at say

$$(\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_2} a_{x_2}^{-1} \gamma_{\bar{f}_1} a_{x_1}^{-1} g)v.$$

Then  $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g$  must fix  $v$ , say  $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g = a_v \in G_v$ .

We now have

$$g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v$$

and we somewhat optimistically define

$$\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R.$$

#### 4. AN INEFFICIENT CHOICE

Is  $\psi$  well defined? The geodesic from  $v$  to  $gv$  is certainly unique, as is the first point  $x_1$  where it leaves  $T$  and its first edge  $e_m$  outside  $T$ . Both the edge  $e^1$  and the group element  $\gamma_{f_1}$  are now determined by our original construction. The only ambiguity at this stage is the choice of the element  $a_{x_1} \in G_{x_1}$  which maps  $e^1$  to  $e_m$ . A different choice  $b_{x_1}$  will give a path from  $z_1$  to  $(\gamma_{\bar{f}_1} b_{x_1}^{-1} g)v$  which leaves  $T$  for the first time at say  $y_2$ . The first edge outside  $T$  will project to an edge  $f'_2$  of  $X/G$  and so on until eventually we have  $g$  expressed as

$$g = b_{x_1} \gamma_{f_1} b_{y_2} \gamma_{f'_2} \dots b_{y_s} \gamma_{f'_s} b_v.$$

We must show that  $a_{x_1} \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v$  and  $b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v$  determine the same left coset of  $R$  in  $(*G_w)*F$ .

Agree to select  $a_{x_1}$  from  $G_{x_1}$  so that the tail of the resulting path is as long as possible. Continue in this way selecting  $a_{x_2}, a_{x_3} \dots$  so as to maximise the length of the tail at each stage. We shall compare any other set of choices with this rather inefficient selection.

Both  $a_{x_1}$  and  $b_{x_1}$  map  $e^1$  to  $e_m$ , so  $c = a_{x_1}^{-1} b_{x_1}$  must fix  $e^1$ . Also, due to our particular selection of  $a_{x_1}$ , the geodesic from  $z_1$  to  $x_2$  is left fixed by  $\gamma_{\bar{f}_1} c \gamma_{f_1}$ . Therefore