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§ 5. THE TRACE

The fundamental idea of V. Jones which led him to the definition of his original one-variable polynomial is the construction of the trace. Originally, V. Jones used algebras which are quotients of the algebras H_n . The lifting of the trace to the Hecke algebras H_n was observed by A. Ocneanu.

The trace will commute with the inclusion $H_n \rightarrow H_{n+1}$ and therefore yield a trace on the direct limit of the H_n 's. (Compare with the discussion in § 12.)

THEOREM. *Let K be a field and let $q, z \in K$ be two elements of K . Let H_n be the Hecke algebra over K corresponding to q . There exists a trace $\text{Tr}: H_n \rightarrow K$ compatible with the inclusion $H_n \rightarrow H_{n+1}$, i.e. the diagram*

$$\begin{array}{ccc} H_n & \xrightarrow{\quad} & H_{n+1} \\ & \searrow \text{Tr} & \swarrow \text{Tr} \\ & & K \end{array}$$

commutes, and such that

- (1) $\text{Tr}(1) = 1$,
- (2) Tr is K -linear and $\text{Tr}(ab) = \text{Tr}(ba)$,
- (3) If $a, b \in H_n$, then $\text{Tr}(aT_n b) = z\text{Tr}(ab)$.

Notice that the last property enables us to calculate $\text{Tr}(x)$ for an arbitrary $x \in H_n$ by using the fact that monomials in normal form generate H_n over K . For instance,

$$\begin{aligned} \text{Tr}(T_1) &= z, \\ \text{Tr}(T_1 T_2) &= \text{Tr}(T_2 T_1) = z^2, \\ \text{Tr}(T_1 T_2 T_1) &= z\text{Tr}(T_1^2) = z((q-1)z + q). \end{aligned}$$

Proof. The K -linear map $\text{Tr}: H_{n+1} \rightarrow K$ is defined by induction on n , using the structure lemma of § 4 (Proposition 4.1):

$$\varphi: H_n \oplus H_n \otimes_{H_{n-1}} H_n \xrightarrow{\sim} H_{n+1}.$$

Starting with $\text{Tr}: H_0 = K \rightarrow K$ the identity, one defines $\text{Tr}: H_{n+1} \rightarrow K$ by $\text{Tr}(x) = \text{Tr}(a) + \sum_i z \text{Tr}(b_i c_i)$, if $\varphi(a + \sum_i b_i \otimes c_i) = x$.

It is clear that if $a, b \in H_n$, then

$$\text{Tr}(aT_n b) = z \text{Tr}(ab),$$

since $\varphi(a \otimes b) = aT_n b$.

The only statement to be proved is then:

$$\text{Tr}(xy) = \text{Tr}(yx) \quad \text{for all } x, y \in H_{n+1}.$$

This is proved by induction on n .

We may assume that x and y are monomials containing T_n at most once.

If y does not contain T_n at all, then writing $x = x' T_n x''$, where x', x'' are monomials in T_1, \dots, T_{n-1} , one has

$$\text{Tr}(xy) = z \text{Tr}(x' x'' y) = z \text{Tr}(y x' x'') = \text{Tr}(y x' T_n x'') = \text{Tr}(y x).$$

If y contains T_n , it suffices to check the case where $x = a T_n b$ and $y = T_n$, as is easily verified. (Here $a, b \in H_n$.)

There are various cases depending on whether or not the elements a and b actually contain T_{n-1} . The worst case is the one in which $a = a' T_{n-1} a''$, $b = b' T_{n-1} b''$ with a', a'', b', b'' belonging to H_{n-1} . We have then

$$\begin{aligned} \text{Tr}(a T_n b T_n) &= z((q-1)\text{Tr}(ab) + q\text{Tr}(ab' b'')) \\ \text{Tr}(T_n a T_n b) &= z((q-1)\text{Tr}(ab) + q\text{Tr}(a' a'' b)). \end{aligned}$$

But

$$\text{Tr}(ab' b'') = \text{Tr}(a' T_{n-1} a'' b' b'') = z \text{Tr}(a' a'' b' b''),$$

and

$$\text{Tr}(a' a'' b) = \text{Tr}(a' a'' b' T_{n-1} b'') = z \text{Tr}(a' a'' b' b'').$$

Hence,

$$\text{Tr}(a T_n b T_n) = \text{Tr}(T_n a T_n b)$$

as desired.