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SOME CHARACTERIZATIONS OF COXETER GROUPS

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ABSTRACT: The aim of this note is to compile together various characterizations of Coxeter groups. Some of these are well-known, some are not-so-well-known and some are entirely new. The motivation behind the new ones is explained in the introduction.

§ 1. INTRODUCTION

Coxeter groups originally arose as group of symmetries of various geometrical objects. These became the center of activity in Lie Theory because of fundamental work of E. Cartan, H. Weyl and others regarding the structure of semi-simple Lie algebras. A little later on, Coxeter gave a complete classification of finite groups generated by reflections which included the Weyl groups, the dihedral groups and two sporadic groups (H_3 and H_4). In doing so, he gave a presentation of these groups which then led to other families of groups which have similar presentations. These include the affine Weyl groups and this is motivation enough to develope the theory of general Coxeter groups. Such a study was initiated around late 60's and an important characterization in terms of the so-called exchange condition was given (cf. [B], [S]). In recent years, this theory has been further developed and a lot of important work has been done. A major part of this work is in connection with the Bruhat ordering and its role in various different contexts in Lie theory ([K-L]). Another object under investigation is the so-called root-system of a general Coxeter group. It is seen that a number of properties of Coxeter groups can be derived using these root-systems ([D]). In an attempt to understand the role of the above two concepts in this theory, the author found out that these two

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properties characterize Coxeter groups. It therefore seems worthwhile to compile together various characterizations of Coxeter groups. This is done in § 2. A part of it is of expository nature though our proofs for the well-known characterizations are somewhat more direct.

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§ 2. Main Theorem

Let W be a group generated by a set S of involutary generators (i.e. order $s = 2\ \forall s \in S$). One then has the notion of the length $l(w)$ of an element $w \in W$ viz. the least integer k such that $w = s_1 \ldots s_k$ with $s_i \in S$. Further, such an expression is called a reduced expression. We then have the following:

Main Theorem. Let W, S be as above. Then the following conditions are equivalent:

1) *Coxeter condition:* If $\tilde{W}$ is the free group generated by a copy $\tilde{S}$ of S subject to relations $(\tilde{s})^2 = \text{id}\ \forall s \in S$ and $\eta : \tilde{W} \to W$ is the canonical map, then Ker η is generated as a normal subgroup by elements of the type:

$\{(\tilde{s}_1\tilde{s}_2)^{m_{s_1,s_2}}, s_1 \neq s_2 \in S, m_{s_1,s_2} \geqslant 2\}$ i.e. $<S \mid s^2 = \text{id}\ \forall s \in S, (s_1s_2)^{m_{s_1,s_2}} = \text{id}$

for some pairs $s_1 \neq s_2$ in $S>$ is a presentation of W. (Note that the above relations *may not* involve *all* pairs $s_1 \neq s_2$).

2) *Root-system condition:* There exists a representation V of W over $\mathbf{R}$, a W-invariant set Φ of non-zero vectors in V which is symmetric (i.e. $\Phi = -\Phi$) and a subset $\{e_s \mid s \in S\}$ of Φ such that the following conditions are satisfied.

(i) Every $\phi \in \Phi$ can be written as $\sum_{s \in S} a_s e_s$ with either all $a_s \geqslant 0$ or all $a_s \leqslant 0$, but not in both ways.

(Accordingly, we write $\phi > 0$ or $\phi < 0$.)

(ii) $e_s \in \Phi$, $s(e_s) < 0$ and $s(\phi) > 0$ for all $\phi > 0$, $\phi \neq e_s$.

(iii) If $w \in W$, $s, s' \in S$ are such that $w(e_{s'}) = e_s$. Then $ws'w^{-1} = s$.