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It will be sufficient to look only at the symmetries of  $H$  which take the fibre  $L_0 = \{(u, 0)\}$  to itself, and hence are of the form  $(u, v) \mapsto (A(u), B(v))$ . We already know that there must be a  $C \in SO(8)$  such that  $B(mu) = C(m) A(u)$  for all  $m, u \in Ca$ . To show that  $G$  is a nontrivial double covering of  $SO(9)$ , we must find a loop of  $C$ 's which lifts to a non-loop of  $(A, B)$ 's.

This can be done by using the Moufang identities, just as in the proof of the Triality Principle. Recall from that proof that if  $x$  is an imaginary Cayley number of unit length, then  $A = L_x$ ,  $B = -L_x$  and  $C = L_x R_x$  "works", that is,  $-L_x(mu) = L_x R_x(m) L_x(u)$ . Now let  $x$  describe a semi-circular path in the  $i, j$ -plane from  $i$  to  $-i$ . At the beginning of the path,  $C(m) = imi$ , while at the end of the path  $C(m) = (-i)m(-i) = imi$ . Thus  $C$  describes a loop in  $SO(8)$ . At the beginning of the path,  $(A(u), B(v)) = (iu, -iv)$ , while at the end  $(A(u), B(v)) = (-iu, iv)$ . Hence  $(A, B)$  describes a non-loop in  $G$ . Thus  $G$  is the non-trivial double covering  $\text{Spin}(9)$  of  $SO(9)$ . QED

Here is a further indication of the extent of symmetry of the Hopf fibration  $H: S^7 \hookrightarrow S^{15} \rightarrow S^8$ . Orient the fibres.

PROPOSITION 7.10. *Let  $P$  and  $Q$  be any two fibres of  $H$ . Then a preassigned orientation preserving rigid motion of  $P$  onto  $Q$  can be extended to a symmetry of  $H$ . In particular, the symmetries act transitively on  $S^{15}$ .*

By Lemma 7.7, the symmetries act transitively on fibres, so we may take  $P = Q = L_0$ . To preassign an orientation preserving rigid motion of  $L_0$  onto itself is to preassign the map  $A \in SO(8)$  in the Triality Principle, which then promises the desired symmetry of  $H$ . QED

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