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THE VARIETY OF MODULI OF FOLIATIONS ON A COMPLEX SPACE

by Hans-Jörg REIFFEN

§ 0. Let X be a complex space. By X_s we denote the singular locus of X. Let $\mathscr{F}$ be a coherent analytic sheaf on X. By $S(\mathscr{F})$ we denote the singular locus of $\mathscr{F}$, i.e. the set of points, where $\mathscr{F}$ is not free. $S(\mathscr{F})$ is an analytic subset of X. If X is reduced then $S(\mathscr{F})$ is thin. For a coherent subsheaf $\mathscr{F}'$ of $\mathscr{F}$ we set $S(\mathscr{F} : \mathscr{F}') := S(\mathscr{F}) \cup S(\mathscr{F}/\mathscr{F}')$.

In the following let X be a reduced complex space. If X' is an irreducible component of X, then we denote by $r(\mathscr{F}, X')$ the rank of $\mathscr{F}$ on $X' \backslash (X_s \cup S(\mathscr{F}))$. Ω resp. Θ is the sheaf of holomorphic 1-forms resp. of vector fields on X. We have $S(\Omega) = X_s$.

0.1 DEFINITION. Let Ω' be a coherent subsheaf of Ω and X' an irreducible component of X. Ω' is a X'-foliation, iff there is a thin analytic subset A of X', $X' \cap X_s \subset A$, such that $\Omega' \mid X' \backslash A$ defines a regular foliation. Ω' is a foliation, iff Ω' is a X'-foliation for every irreducible component X' of X.

In a joint work with G. Bohnhorst (comp. [B/R]) I have developed a general theory of foliations. The coherent foliations as they are introduced in [B/R] correspond to those foliations used in this paper, which are full subsheaves of Ω. We call $\operatorname{codim}_{X'} \Omega' := r(\Omega'; X')$ the codimension of the X'-foliation.

The following homomorphism of sheaves (of groups) is important

$$\delta^p : \Omega^{p+1} \to \overset{p+2}{\wedge} \Omega, \ \delta^p(\omega^0; \omega^1, \ldots, \omega^p) := d\omega^0 \wedge \omega^1 \wedge \ldots \wedge \omega^p .$$

0.2 REMARK. Let Ω' be a coherent subsheaf of Ω. Then the following are equivalent:

(1) Ω' is a X'-foliation of codimension p;

(2) $p = r(\Omega'; X')$, $\delta^p \mid (\Omega' \mid X' \backslash X_s)^{p+1} = 0$;

(3) $p = r(\Omega'; X')$, there is a point $x \in X' \backslash X_s$, such that $\delta^p((\Omega'_x)^{p+1}) = 0$.

For the following compilation of definitions and results the paper by Douady [Dou] is a good reference.

In the following let Y, Z be further complex spaces, that are not necessarily reduced. If $\mathscr{E}$ is a coherent analytic sheaf on Y and $f: Z \to Y$ a holomorphic mapping then we denote by $f^*\mathscr{E}$ the analytic inverse image of $\mathscr{E}$ on Z.

Let $\mathscr{E}$ be a coherent analytic sheaf on $Y \times X$. For $y \in Y$ we denote by $\mathscr{E}(y)$ the analytic restriction of $\mathscr{E}$ on $X = y \times X$; $\mathscr{E}(y)$ is the analytic inverse image of $\mathscr{E}$ to the injection $X = y \times X \hookrightarrow Y \times X$. If $f: Z \to Y$ is a holomorphic mapping then we set $\mathscr{E}_Z := (f \times \mathrm{id}_X)^*\mathscr{E}$. We have $\mathscr{E}_Z(z) = \mathscr{E}(f(z))$ for $z \in Z$.

Let the coherent analytic sheaf $\mathscr{E}$ on $Y \times X$ be Y-flat, then $r(\mathscr{E}(y), X')$ is locally constant on Y for every irreducible component X' of X. Therefore

$$R^p_{X'}(Y, \mathscr{E}) := \{y \in Y : r(\mathscr{E}(y), X') = p\}$$

is an open and closed subset of Y for every p. Let $\check{\mathscr{R}}$ be a coherent subsheaf of $\mathscr{E}$, so that $\mathscr{R} := \mathscr{E}/\check{\mathscr{R}}$ is also Y-flat. Then $\check{\mathscr{R}}$ is Y-flat too and we have $\mathscr{R}(y) = \mathscr{E}(y)/\check{\mathscr{R}}(y)$ in a natural way. Let $f: Z \to Y$ be a holomorphic mapping. Then $\mathscr{E}_Z$ and $\mathscr{R}_Z$ are Z-flat and we have $\mathscr{R}_Z = \mathscr{E}_Z/\check{\mathscr{R}}_Z$ in a natural way.

Let $\mathscr{F}$ be a coherent analytic sheaf on X. Then the analytic inverse image $\mathscr{F}^Y := \pi^*\mathscr{F}$ to the projection $\pi: Y \times X \to X$ is Y-flat and we have $\mathscr{F}^Y(y) = \mathscr{F}$ in a natural way. In the following let $\check{\mathscr{R}}$ be a coherent subsheaf of Ω^Y, such that $\mathscr{R} := \Omega^Y/\check{\mathscr{R}}$ is Y-flat. Let X' be an irreducible component of X. We set:

$$F^p_{X'}(Y, \check{\mathscr{R}}) := \{y \in Y : \check{\mathscr{R}}(y) \text{ is a } X'\text{-foliation of codimension } p\},$$

$$F_{X'}(Y, \check{\mathscr{R}}) := \{y \in Y : \check{\mathscr{R}}(y) \text{ is a } X'\text{-foliation}\},$$

$$F(Y, \check{\mathscr{R}}) := \{y \in Y : \check{\mathscr{R}}(y) \text{ is a foliation}\}.$$

We will show:

0.3 THEOREM. *$F^p_{X'}(Y, \check{\mathscr{R}})$ is an analytic subset of Y.*

By 0.3 we get:

0.4 COROLLARY.

(1) *$F_{X'}(Y, \check{\mathscr{R}})$ is an analytic subset of Y;*

(2) *$F(Y, \check{\mathscr{R}})$ is an analytic subset of Y.*

Because $\mathscr{R}$ is Y-flat we get the following:

0.5 Remark. $S(\Omega^Y : \check{\mathscr{R}}) \cap (y \times X) = S(\Omega : \check{\mathscr{R}}(y))$.

In 0.5 we identify $X = y \times X$.

Let $f : Z \to Y$ be a holomorphic mapping. Then $\Omega^Y_Z = \Omega^Z$ and we obviously get:

0.6 Remark. $F^p_{X'}(Z, \check{\mathscr{R}}_Z) = f^{-1}(F^p_{X'}(Y, \check{\mathscr{R}}))$.

$F^p_{X'}(Y, \check{\mathscr{R}})$ is an analytic subset of $R^p_{X'}(Y, \check{\mathscr{R}})$. Using the homomorphism δ^p we construct a canonical complex structure on $F^p_{X'}(Y, \check{\mathscr{R}})$ and show:

0.7 Theorem. *The holomorphic mapping $f : Z \to Y$ induces a holomorphic mapping $F^p_{X'}(Z, \check{\mathscr{R}}_Z) \to F^p_{X'}(Y, \check{\mathscr{R}})$ related to the canonical complex structures.*

We can apply $F(Y, \check{\mathscr{R}})$ with the structure given by the spaces $F^p_{X'}(Y, \check{\mathscr{R}})$. We get the following (comp. [Dou]):

0.8 Corollary. *Let X be compact and let H be the Douady space of Ω, $\check{\mathscr{R}}$ the universal Douady sheaf of Ω. Then the complex subspace $F(H, \check{\mathscr{R}})$ of H has the following property:*

If Z is a complex space and $\mathscr{S} = \Omega^Z/\check{\mathscr{S}}$ a coherent and Z-flat sheaf, then the unique holomorphic mapping $f : Z \to H$ with $\mathscr{S} = \check{\mathscr{R}}_Z$ factorizes over $F(Z, \check{\mathscr{S}})$, $F(H, \check{\mathscr{R}})$.

In this sense $F(H, \check{\mathscr{R}})$ is a moduli space of the foliations on X.

0.9 Proposition. *Let X be a connected compact manifold. Then the following sets are complements of analytic subsets in $F(Y, \check{\mathscr{R}})$:*

$F_r(Y, \check{\mathscr{R}}) := \{y \in Y : \check{\mathscr{R}}(y) \text{ is a regular foliation}\}$,

$F_f(Y, \check{\mathscr{R}}) := \{y \in F(Y, \check{\mathscr{R}}) : \check{\mathscr{R}}(y) \text{ is locally free}\}$,

${}^kF(Y, \check{\mathscr{R}}) := \{y \in F(Y, \check{\mathscr{R}}) : \operatorname{codim} S(\Omega : \check{\mathscr{R}}(y)) \geqslant k\}$.

$F_f(Y, \check{\mathscr{R}}) \cap {}^2F(Y, \check{\mathscr{R}})$ is the set of points $y \in Y$ such that $\check{\mathscr{R}}(y)$ is a free foliation in the sense of [B/R]. By the theorem of Frobenius-Malgrange all foliations $\check{\mathscr{R}}(y)$, $y \in F_f(Y, \check{\mathscr{R}}) \cap {}^3F(Y, \check{\mathscr{R}})$ are locally integrable.

In an earlier version of this paper ([Re]) I gave a proof of 0.3 by Banach analytic technics. With these technics G. Pourcin has gotten results similar to those in this paper ([Pou]). Similar results were also proved by X. Gomez-Mont ([GM]). He considers foliations defined by vector fields; his technics are closer to those of this paper. The advantage of my approach is the simplicity. For 0.3, the canonical complex structure on $F^p_{X'}(Y, \check{\mathscr{R}})$, $F(Y, \check{\mathscr{R}})$ and 0.7 we need no compactness.

§ 1. We prove 0.3: Because 0.3 is a local theorem related to Y and because of 0.2 we may assume the following:

Y is an analytic subspace of a polycylinder $P = \{s \in \mathbf{C} : |s| < \sigma\}^m$ in $\mathbf{C}^m$ defined by an ideal sheaf $\mathscr{I}$. We denote the coordinates of $\mathbf{C}^m$ by $y = (y_1, ..., y_m)$. $Y = R^p_{X'}(Y, \check{\mathscr{R}})$.

X is a polycylinder $\{t \in \mathbf{C} : |t| < \tau\}^n$ in $\mathbf{C}^n$. We denote the coordinates of $\mathbf{C}^n$ by $x = (x_1, ..., x_n)$. $S(\Omega^Y : \check{\mathscr{R}}) = \emptyset$.

We can interprete $(\overset{r}{\wedge}\Omega)^P$ as a submodule of $\overset{r}{\wedge}\Omega_{P \times X}$: $(\overset{r}{\wedge}\Omega)^P$ consists of all r-forms η of the form $\eta = \sum_{1 \leqslant \nu_1 < \dots < \nu_r \leqslant n} \eta_{\nu_1 \dots \nu_r}(y, x) dx_{\nu_1} \wedge \dots \wedge dx_{\nu_r}$.
We notice this by $\eta = \eta(y, x; dx)$. We get $(\overset{r}{\wedge}\Omega)^Y = (\overset{r}{\wedge}\Omega)^P / \mathscr{I} . (\overset{r}{\wedge}\Omega)^P$.
We may assume, that there are forms $\eta^1, ..., \eta^q \in \Gamma(P \times X, \Omega^P)$ generating $\check{\mathscr{R}}$ everywhere. We shorten $F := F^p_{X'}(Y, \check{\mathscr{R}})$. For $y^0 \in Y$ we get:

$$y^0 \in F \Leftrightarrow \delta^p(\eta^{j_0}(y^0, x; dx); \eta^{j_1}(y^0, x; dx), ..., \eta^{j_p}(y^0, x; dx)) = 0$$
$$\text{for all} \quad 1 \leqslant j_0, j_1, ..., j_p \leqslant q .$$

We consider an arbitrary system $\omega^0, \omega^1, ..., \omega^p \in \Gamma(P \times X, \Omega^P)$. By fixing y we define

$$\vartheta^p(\omega^0; \omega^1, ..., \omega^p)$$
$$:= \delta^p(\omega^0(y, x; dx); \omega^1(y, x; dx), ..., \omega^p(y, x; dx)) \in \Gamma(P \times X, (\overset{p+2}{\wedge}\Omega)^P) .$$

We have

$$\vartheta^p(\omega^0; \omega^1, ..., \omega^p) = \sum_{1 \leqslant \nu_1 < \dots < \nu_{p+2} \leqslant n} A_{\nu_1 \dots \nu_{p+2}} dx_{\nu_1} \wedge \dots \wedge dx_{\nu_{p+2}}$$

with holomorphic coefficients

$$A_{\nu_1 \dots \nu_{p+2}} = A_{\nu_1 \dots \nu_{p+2}}(\omega^0; ..., \omega^p) \in \Gamma(P \times X, \mathcal{O}_{P \times X}) .$$

We form the serial expansions

$$A_{\nu_1 \dots \nu_{p+2}} = \sum_{l \in \mathbf{N}^n} A_{\nu_1 \dots \nu_{p+2}; l} x^l$$

with holomorphic coefficients $A_{\nu_1 \dots \nu_{p+2}; l} = A_{\nu_1 \dots \nu_{p+2}; l}(\omega^0, ..., \omega^p) \in \Gamma(P, \mathcal{O}_P)$.
With these notations we get:

The set F is defined on Y by the following (infinite) system of holomorphic functions:

$$A_{\nu_1 \dots \nu_{p+2}; l}(\eta^{j_0}; \eta^{j_1}, ..., \eta^{j_p}) ;$$
$$1 \leqslant j_0, j_1, ..., j_p \leqslant q, \ 1 \leqslant \nu_1 < \dots < \nu_{p+2} \leqslant n, \ l \in \mathbf{N}^n .$$

F is an analytic subset of Y.

§ 2. We refine the considerations of § 1. We can interprete ϑ^p as a homomorphism of sheaves (of groups) $\vartheta^p : (\Omega^P)^{p+1} \to (\overset{p+2}{\wedge} \Omega)^P$. Obviously ϑ^p induces a homomorphism $\vartheta^p_Y : (\Omega^Y)^{p+1} \to (\overset{p+2}{\wedge})^Y$, which does not depend on the representation of Y as a subspace of a number space. Let $\mathscr{J}$ be the ideal sheaf on P, defined by $\mathscr{J}$ and the functions $A_{\nu_1 \ldots \nu_{p+2};l}(\eta^{j_0}; \eta^{j_1}, \ldots, \eta^{j_p})$ (comp. [Fr]). We apply F with the structure sheaf $\mathscr{H} := \mathscr{O}_P/\mathscr{J}$ and consider the natural injection $\iota : F \to Y$.

2.1 Proposition. *$\mathscr{H}$ is the maximal complex structure on F, such that $\vartheta^p_{(F,\mathscr{H})} \mid (\check{\mathscr{R}}_{(F,\mathscr{H})})^{p+1} = 0$, i.e.*

(1) $\vartheta^p_{(F,\mathscr{H})} \mid (\check{\mathscr{R}}_{(F,\mathscr{H})})^{p+1} = 0$;

(2) *if $(F, \tilde{\mathscr{H}})$ is an analytic subspace of Y, such that $\vartheta^p_{(F,\tilde{\mathscr{H}})} \mid (\check{\mathscr{R}}_{(F,\tilde{\mathscr{H}})})^{p+1} = 0$, then $(F, \tilde{\mathscr{H}})$ is an analytic subspace of $(F, \mathscr{H})$.*

Proof. (1) Let $\omega^0, \omega^1, \ldots, \omega^p \in \Gamma(P \times X, \Omega)$ induce elements of $\Gamma(F \times X, \check{\mathscr{R}}_H)$. We may assume, that there are representations

$$\omega^k = \sum_{j=1}^{q} a_{kj}\, \eta^j \bmod \Gamma(P \times X, \mathscr{J} . \Omega^P) .$$

Because of

$$\eta^{j_0} \wedge \eta^{j_1} \wedge \ldots \wedge \eta^{j_p} \in \Gamma(P \times X, \mathscr{J} . (\overset{p+1}{\wedge} \Omega)^P)$$

we get mod $\Gamma(P \times X, \mathscr{J} . (\overset{p+2}{\wedge} \Omega)^P)$:

$$\vartheta^p(\omega^0; \omega^1, \ldots, \omega^p) = \sum_{1 \leq j_0, \ldots, j_p \leq q} a_{0j_0} \cdots a_{pj_p}\, \vartheta^p(\eta^{j_0}; \eta^{j_1}, \ldots, \eta^{j_p}) .$$

Therefore we get

$$A_{\nu_1 \ldots \nu_{p+2};l}(\omega^0; \omega^1, \ldots, \omega^p) \in \Gamma(P, \mathscr{J}),\ \vartheta^p(\omega^0; \omega, \ldots, \omega^p) \in \Gamma(P \times X, \mathscr{J} . (\overset{p+2}{\wedge} \Omega)^P) .$$

(2) Let $\tilde{\mathscr{J}}$ be the ideal sheaf of $\mathscr{O}_P$ defining $\tilde{\mathscr{H}}$. We show $\mathscr{J} \subset \tilde{\mathscr{J}}$. We have

$$\vartheta^p(\eta^{j_0}; \eta^{j_1}, \ldots, \eta^{j_p}) \in \Gamma(P \times X, \tilde{\mathscr{J}} . (\overset{p+2}{\wedge} \Omega)^P)$$

and therefore

$$A_{\nu_1 \ldots \nu_{p+2},l}(\eta^{j_0}; \eta^{j_1}, \ldots, \eta^{j_p}) \in \Gamma(P, \tilde{\mathscr{J}}) .$$

Now we consider an arbitrary reduced complex space X and an arbitrary complex space Y. We again set $F := F^p_{X'}(Y, \check{\mathscr{R}})$.

Let $y^0 \in F$. We consider $X'(y^0) := X' \backslash S(\Omega : \check{\mathscr{R}}(y^0))$. If $x^0 \in X'(y^0)$ then we can realize the situation of § 1 related to open neighbourhoods V_0 of y^0 and U_0 of x^0. Let $\mathscr{H}(x^0)$ denote the structure sheaf of $V_0 \cap F$ according to 2.1. We get $\mathscr{H}(x^0)_{y^0} = \mathscr{H}(x)_{y^0}$ for every $x \in U_0$. Because $X'(y^0)$ is connected, we get $\mathscr{H}(x^0)_{y^0} = \mathscr{H}(x)_{y^0}$ for every $x \in X'(y^0)$. Let $\mathscr{H}$ be the structure sheaf on F defined by $\mathscr{H}_y := \mathscr{H}(x)_y$, $x \in X'(y)$.

If $\tilde{\mathscr{H}}$ is any structure sheaf on F such that $(F, \tilde{\mathscr{H}})$ is an analytic subspace of Y we get by 0.5

$$S(\Omega^{(F, \tilde{\mathscr{H}})} : \check{\mathscr{R}}_{(F, \tilde{\mathscr{H}})}) = S(\Omega^Y : \check{\mathscr{R}}) \cap (F \times X) .$$

We set

$$(F \times X)^0 := (F \times X) \backslash S(\Omega^Y : \check{\mathscr{R}}) .$$

By 2.1 we get:

2.2 PROPOSITION. *$\mathscr{H}$ is the maximal complex structure on F, such that* $\vartheta^p_{(F,\mathscr{H})} \mid (\check{\mathscr{R}}_{(F,\mathscr{H})} \mid (F \times X)^0)^{p+1} = 0$.

We prove 0.7: We may assume that Z, X and Y, X fulfill the assumptions of § 1 simultaneously. We denote the objects related to Z by the index Z. Further we may assume, that f is given by a holomorphic mapping $g: P_Z \to P$ with $g^*\mathscr{I} \subset \mathscr{I}_Z$. We consider $\omega^j(z, x; dx) := \eta^j(g(z), x; dx)$. Then ω^j generates an element of $\Gamma(Z \times X, \check{\mathscr{R}}_Z)$ and we get

$$\begin{aligned} &A_{\nu_1 \ldots \nu_{p+2};l}(\eta^{j_0}; \eta^{j_1}, \ldots, \eta^{j_p}) \circ (f \times \mathrm{id}_X) \\ &= A_{\nu_1 \ldots \nu_{p+2};l}(\omega^{j_0}; \omega^{j_1}, \ldots, \omega^{j_p}) \in \Gamma(P_Z, \mathscr{I}_Z) . \end{aligned}$$

We prove 0.9: Let $\pi: Y \times X \to Y$ be the projection then $\pi(S(\Omega^Y : \check{\mathscr{R}}))$ and $\pi(S(\check{\mathscr{R}}))$ are analytic subsets of Y. $\{(y, x) \in Y \times X : \dim_{(y,x)}(S(\Omega^Y : \check{\mathscr{R}}) \cap (y \times X)) > \dim X - k\}$ is analytic and therefore $\{y \in Y : \operatorname{codim} S(\Omega : \check{\mathscr{R}}(y)) < k\}$ too.

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