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5. *R*-MATRICES AND INTERTWINING OPERATORS

In this section we shall prove that, after a trivial twisting, the intertwining operators between certain representations of Yangians provide rational solutions of the quantum Yang-Baxter equation. Recall that, if  $V$  is any representation of  $Y = Y(\mathfrak{sl}_2)$ , then, for any  $a \in \mathbf{C}$ , we denote by  $V(a)$  its pull-back by the automorphism  $\tau_a$  of  $Y$  defined in Proposition 2.5.

PROPOSITION 5.1. *Let  $V, W$  be irreducible finite-dimensional representations of  $Y$  with highest weight vectors  $\Omega_V, \Omega_W$  and let  $a, b \in \mathbf{C}$ . Then:*

(a) *the tensor products  $V(a) \otimes W(b)$  and  $W(b) \otimes V(a)$  are irreducible and isomorphic except for a finite set of values  $S(V, W)$  of  $a - b$ ;*

(b) *the unique intertwining operator*

$$I(V, a; W, b): W(b) \otimes V(a) \rightarrow V(a) \otimes W(b)$$

*which maps  $\Omega_W \otimes \Omega_V$  to  $\Omega_V \otimes \Omega_W$  is a rational function of  $a - b$  with values in  $\text{Hom}(W \otimes V, V \otimes W)$ .*

*Proof.* Part (a) follows immediately from Proposition 4.2 and Corollary 4.7. For part (b), we need the following lemma.

LEMMA 5.2. *Let  $V, W$  be representations of  $Y$  and let  $a \in \mathbf{C}$ .*

(a) *If  $V$  is irreducible, so is  $V(a)$ .*

(b) *If  $I: V \rightarrow W$  is an isomorphism of representations of  $Y$ , so is  $I: V(a) \rightarrow W(a)$ .*

*Proof of lemma.* Part (a) follows from the definition of  $V(a)$ . For part (b), we must show that  $I$  commutes with the action of  $x$  and  $J(x)$  on  $V(a)$  and  $W(a)$ , for all  $x \in \mathfrak{sl}_2$ . But this is clear, since the action of  $x$  is the same as that on  $V$  and  $W$ , and that of  $J(x)$  is the same as that of  $J(x) + ax$  on  $V$  and  $W$ .

Returning to the proof of Proposition 5.1, it follows from the lemma that  $I(V, a; W, b)$  is a function of  $a - b$ , so it suffices to consider the case  $b = 0$ . For any  $a \in \mathbf{C}$  which does not belong to the finite set  $S(V, W)$ , there is a unique isomorphism

$$I(V, a; W, 0) \equiv I(a): W \otimes V(a) \rightarrow V(a) \otimes W$$

of representations of  $Y$  such that

$$(5.3) \quad I(a) (\Omega_W \otimes \Omega_V) = \Omega_V \otimes \Omega_W .$$

Choose bases of  $V \otimes W$  and  $W \otimes V$  and let  $\{I_\lambda\}$  be a basis of  $\mathfrak{sl}_2$ ; write  $I(a)$  also for the matrix of  $I(a)$  with respect to these bases. Let  $A_\lambda, B_\lambda$  be the matrices of  $I_\lambda$  and  $J(I_\lambda)$  acting on  $W \otimes V(a)$ ; and let  $A'_\lambda$  and  $B'_\lambda$  refer similarly to  $V(a) \otimes W$ . Then,  $I(a)$  commutes with the action of  $Y$  if and only if  $I(a)$  satisfies the following system of homogeneous linear equations:

$$A_\lambda I(a) = I(a) A'_\lambda, \quad B_\lambda I(a) = I(a) B'_\lambda, \quad \text{for all } \lambda .$$

We know that, if  $a \notin S(V, W)$ , these equations have a unique solution satisfying equation (5.3). By elementary linear algebra, the solution is a rational function of the entries of the matrices  $A_\lambda, A'_\lambda, B_\lambda, B'_\lambda$ . Since  $A_\lambda, A'_\lambda$  are independent of  $a$  and  $B_\lambda, B'_\lambda$  are linear in  $a$ , the result follows.

*Definition 5.4.* Let  $V$  be a finite-dimensional irreducible representation of  $Y$ . Then, the  $R$ -matrix associated to  $V$  is the function  $R(a-b)$  with values in  $\text{End}(V \otimes V)$  given by

$$R(a-b) = I(V, a; V, b) \sigma ,$$

where  $\sigma \in \text{End}(V \otimes V)$  is the switch of the two factors.

**THEOREM 5.5.** *Let  $V$  be a finite-dimensional irreducible representation of  $Y$ . Then the  $R$ -matrix associated to  $V$  is a rational solution of the quantum Yang-Baxter equation:*

$$(5.6) \quad R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) = R^{23}(b-c)R^{13}(a-c)R^{12}(a-b) .$$

*Proof.* We note first some simple commutation relations between the intertwining operator  $I(a-b) \equiv I(V, a; V, b)$  and the switch map  $\sigma$ . For example, we have

$$\sigma^{12} I^{13}(a-c) \sigma^{12} = I^{23}(a-c) .$$

by an easy computation. Similarly,

$$\sigma^{12} \sigma^{13} I^{23}(b-c) \sigma^{13} \sigma^{12} = I^{12}(b-c) .$$

Hence,

$$\begin{aligned} R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) &= I^{12}(a-b)\sigma^{12}I^{13}(a-c)\sigma^{13}I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b)I^{23}(a-c)\sigma^{12}\sigma^{13}I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b)I^{23}(a-c)I^{12}(b-c)\sigma^{12}\sigma^{13}\sigma^{23} . \end{aligned}$$

Similarly,

$$R^{23}(b-c)R^{13}(a-c)R^{12}(a-b) = I^{23}(b-c)I^{12}(a-c)I^{23}(a-b)\sigma^{23}\sigma^{13}\sigma^{12} .$$

Hence, in view of the relation

$$\sigma^{12} \sigma^{13} \sigma^{23} = \sigma^{23} \sigma^{13} \sigma^{12}$$

in the symmetric group on three letters, the equation to be proved is

$$(5.7) \quad I^{12}(a-b)I^{23}(a-c)I^{12}(b-c) = I^{23}(b-c)I^{12}(a-c)I^{23}(a-b) .$$

Note that both sides of equation (5.7) define intertwining operators

$$V(c) \otimes V(b) \otimes V(a) \rightarrow V(a) \otimes V(b) \otimes V(c)$$

which fix the tensor product of the highest weight vectors in  $V$ . Hence, regarded as functions on  $\mathbf{C}^3$  with values in  $\text{End}(V \otimes V \otimes V)$ , they agree on the complement of the set  $S$  of  $(a, b, c) \in \mathbf{C}^3$  where  $V(c) \otimes V(b) \otimes V(a)$  or  $V(a) \otimes V(b) \otimes V(c)$  is reducible. It follows from part (a) of Proposition 5.1 that  $S$  intersects each complex line parallel to one of the axes in  $\mathbf{C}^3$  in at most finitely many points. It is easy to see that the complement of such a set is Zariski dense in  $\mathbf{C}^3$ . Since the two sides of equation (5.7) are rational functions which agree on a Zariski dense set, they are equal.

*Remark.* We have used the following simple fact about intertwining operators. Let  $U, V$  and  $W$  be representations of a Yangian  $Y(\mathfrak{sl}_2)$  and let  $I: U \otimes V \rightarrow V \otimes U$  be an intertwining operator. Then

$$I^{12}: U \otimes V \otimes W \rightarrow V \otimes U \otimes W$$

and

$$I^{23}: W \otimes U \otimes V \rightarrow W \otimes V \otimes U$$

are intertwining operators. While this is obvious enough, it should be noted that

$$I^{13}: U \otimes W \otimes V \rightarrow V \otimes W \otimes U$$

is *not* an intertwining operator in general.

We conclude this general discussion by showing that, up to a sign change in the parameter, the  $R$ -matrix  $R(u)$  we have associated to a representation of  $Y$  is the same as that constructed using the ‘‘universal  $R$ -matrix’’ (see Theorem 3 of [4]). Set

$$\tilde{R}(u) = R(-u) .$$

Then, by Theorem 4 of [4], it suffices to prove that

$$(5.8) \quad P_\lambda^+(a, b)\tilde{R}(b-a) = \tilde{R}(b-a)P_\lambda^-(a, b)$$

where

$$P_\lambda^\pm(a, b) = (\rho \otimes \rho) ((J(I_\lambda) + aI_\lambda) \otimes 1 + 1 \otimes (J(I_\lambda) + bI_\lambda) + \frac{1}{2} [I_\lambda \otimes 1, \Omega]) ,$$

$\rho: Y \rightarrow \text{End}(V)$  is the action of  $Y$  on  $V$  and  $\{I_\lambda\}$  is an orthonormal basis of  $\mathfrak{sl}_2$ . In terms of intertwining operators, equation (5.8) asserts that

$$P_\lambda^+(a, b)I(a-b) = I(a-b)\sigma P_\lambda^-(a, b)\sigma .$$

But it is easy to see that

$$\sigma P_\lambda^-(a, b)\sigma = P_\lambda^+(b, a) .$$

Hence, we must prove that

$$P_\lambda^+(a, b)I(a-b) = I(a-b)P_\lambda^+(b, a) .$$

But this is simply the statement that

$$I(a-b): V(b) \otimes V(a) \rightarrow V(a) \otimes V(b)$$

commutes with the action of  $J(I_\lambda)$ .

We shall now apply these results to compute the  $R$ -matrices associated to every finite-dimensional irreducible representation of  $Y$ . By Theorem 4.11, every such representation is of the form

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k).$$

The intertwining operator

$$I(a-b): V(b) \otimes V(a) \rightarrow V(a) \otimes V(b)$$

can be computed as the product of  $k^2$  intertwining operators of the form  $I(V_m, a; V_n, b)$ , each of which effects an interchange of nearest neighbours. Since such an operator commutes, in particular, with the action of  $\mathfrak{sl}_2$ , it can be written in the form

$$(5.9) \quad I(V_m, a; V_n, b) = \sum_{j=0}^{\min\{m, n\}} c_j P_{m+n-2j} ,$$

where

$$P_{m+n-2j}: V_n \otimes V_m \rightarrow V_m \otimes V_n$$

is the projection onto the irreducible component of

$$V_m \otimes V_n \cong \bigotimes_{j=0}^{\min\{m, n\}} V_{m+n-2j}$$

of type  $V_{m+n-2j}$ . We have  $c_0 = 1$  since  $I(V_m, a; V_n, b)$  preserves the tensor products of the highest weight vectors.

To compute  $I(V_m, a; V_n, b)$ , let  $\Omega_j, j = 0, 1, \dots, \min\{m, n\}$ , be a highest weight vector in  $V_n \otimes V_m$  of weight  $m + n - 2j$ ; then, the vector  $\Omega'_j$  obtained by switching the order of the factors in  $\Omega_j$  is a highest weight vector in  $V_m \otimes V_n$  of the same weight, and we have

$$I(V_m, a; V_n, b) (\Omega_j) = \Omega'_j.$$

Further, it is easy to see that, for  $j > 0$ ,  $(x^+ \otimes 1) \cdot \Omega_j$  is an  $\mathfrak{sl}_2$ -highest weight vector of weight  $m + n - 2j + 2$ ; it is non-zero, since otherwise  $\Omega_j$  would be annihilated by  $x^+ \otimes 1$  and by  $1 \otimes x^+$ , contradicting the assumption  $j > 0$ . Hence, we may assume that

$$(x^+ \otimes 1) \cdot \Omega_j = \Omega_{j-1}$$

for  $j > 0$ . Switching the order of the factors, we have

$$(x^+ \otimes 1) \cdot \Omega'_j = -\Omega'_{j-1}.$$

By Proposition 4.2 (and its proof),  $\Omega_j$  is a  $Y$ -highest weight vector in  $V_n(b) \otimes V_m(a)$  if

$$b - a = \frac{1}{2} (m + n) - j + 1.$$

It follows from the formula for the co-multiplication in Definition 1.1 that, in the representation  $V_n(b) \otimes V_m(a)$ ,

$$J(x^+) \cdot \Omega_j = \left( b - a - \frac{1}{2} (m + n) + j - 1 \right) (x^+ \otimes 1) \cdot \Omega_j,$$

and that in the representation  $V_m(a) \otimes V_n(b)$ ,

$$J(x^+) \cdot \Omega'_j = \left( a - b - \frac{1}{2} (m + n) + j - 1 \right) (x^+ \otimes 1) \cdot \Omega'_j.$$

The equation

$$I(V_m, a; V_n, b) (J(x^+) \cdot \Omega_j) = J(x^+) \cdot (I(V_m, a; V_n, b) \Omega_j)$$

now gives

$$\frac{c_j}{c_{j-1}} = \frac{a - b + \frac{1}{2} (m + n) - j + 1}{a - b - \frac{1}{2} (m + n) - j + 1}.$$

It follows that

$$(5.10) \quad I(V_m, a; V_n, b) = \sum_{j=0}^{\min\{m, n\}} \prod_{i=0}^{j-1} \frac{a - b + \frac{1}{2}(m+n) - i}{a - b - \frac{1}{2}(m+n) + i} P_j .$$

We summarize our results in the following theorem.

**THEOREM 5.11.** *The R-matrix associated to the representation*

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k)$$

of  $Y$  is given by

$$R(a-b) = \left( \prod_{i,j=1}^k I(V_{m_i}, a + a_i; V_{m_j}, b + a_j) \right) \sigma ,$$

where the intertwining operators are given by equation (5.10) and  $\sigma$  is the switch map. The order of the factors in the product is such that the  $(i, j)$ -term appears to the left of the  $(i', j')$ -term iff

$$i > i' \quad \text{or} \quad i = i' \quad \text{and} \quad j < j' .$$

## 6. CONCLUDING REMARKS

Since we have discussed only the Yangian associated to  $\mathfrak{sl}_2$  in this paper, it may be worth-while to indicate the extent to which the results above can be generalized to the Yangian  $Y(\mathfrak{a})$  associated to an arbitrary finite-dimensional complex simple Lie algebra  $\mathfrak{a}$ .

The definition of  $Y(\mathfrak{a})$  is precisely as in (1.1), except of course that  $\{I_\lambda\}$  should be an orthonormal basis of  $\mathfrak{a}$  with respect to some invariant inner product. The formulae

$$\tau_a(x) = x, \quad \tau_a(J(x)) = J(x) + ax,$$

for  $x \in \mathfrak{a}$ , again define a one-parameter group of Hopf algebra automorphisms of  $Y(\mathfrak{a})$ , and the relation, discussed in section 5, between solutions of the quantum Yang-Baxter equation and intertwining operators between tensor products of representations of  $Y(\mathfrak{a})$ , which follows from the existence of the  $\tau_a$ , is also valid in the general case.