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# THE HADAMARD-CARTAN THEOREM IN LOCALLY CONVEX METRIC SPACES

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## 1. INTRODUCTION

M. Gromov has stated the following theorem, which generalizes the classical Hadamard-Cartan theorem from Riemannian manifolds of non-positive sectional curvature to a much richer class of metric spaces in which sectional curvature need not be defined [Gv1, Gv2]:

THEOREM 1 [Gromov]. *A simply connected, complete, locally convex geodesic space is globally convex; hence any two points are joined by a unique geodesic.*

Our interest in Gromov's theorem arose from the wish to apply it to Riemannian manifolds with boundary (see [ABB]). This note gives a proof of the theorem and relates it to work of A. D. Alexandrov, H. Busemann and S. Cohn-Vossen. In this adaptation of the classical proof of the Hadamard-Cartan theorem to the setting of locally convex geodesic spaces, there are two points that require additional attention: finding an appropriate definition of the exponential map, and showing that it is locally surjective. Additionally, for spaces of curvature bounded above in Alexandrov's sense by $K>0$, we show local surjectivity out to length $\pi/\sqrt{K}$ (Theorem 3); since local injectivity is a consequence of Alexandrov's development method, this fully generalizes the classical estimate on conjugate distance. Finally, we extend the Hadamard-Cartan theorem to locally compact geodesic spaces without conjugate points (Theorem 6). We have subsequently learned that our proof of Theorem 1 parallels a sketch proposed by Gromov in oral lectures; we hope that an exposition of these ideas will be useful.

The class of locally convex geodesic spaces includes, for instance, two-dimensional polyhedral surfaces whose simplices are isometric to simplices in a space of constant nonpositive curvature and for which the total angle at each vertex not on the boundary is at least $2\pi$ ([A2]; also see [Gv2, section 4.2] and [Ba] for the $n$-dimensional analogue). Orbifolds modeled on quotients of nonpositively curved spaces and ramified coverings of nonpositively curved spaces also offer many examples. All these are discussed in Gromov's *Hyperbolic Groups* [Gv2], which contains a far-reaching discussion of various types of hyperbolicity (also see [GLP, chapter 1]). Complete Riemannian manifolds with boundary satisfying appropriate interior and boundary curvature conditions are also examples (see [Gv1] and [ABB]). Note that the examples mentioned here allow bifurcation of geodesics; that is, a maximal extension of a geodesic need not be unique.

The earliest generalization of the Hadamard-Cartan theorem to nonriemannian locally convex spaces seems to be due to Busemann [Bu1], in the context of *G-spaces* (see Remark 2 below). The versions due to Busemann [Bu1, Bu2] and Rinow [R] concern spaces for which geodesics do not bifurcate and are infinitely extendible, and which are locally compact and satisfy domain invariance. Note that all of these hypotheses have been eliminated in Theorem 1. For a simple example of a complete, locally convex geodesic space satisfying none of them, take the metric completion of the simply connected covering of the punctured Euclidean or hyperbolic closed disk. In particular, where Busemann and Rinow use the unique and infinite extendibility of geodesics to define the exponential map on the product of $[0, \infty)$ with a metric sphere around $m$, here it is defined as the endpoint map on the space of geodesics starting at $m$, in the uniform metric. Where they use invariance of domain to show that the exponential map is locally surjective, here this is shown to follow from local convexity alone.

*Geodesic spaces* were first considered by Alexandrov [A1], who defined upper curvature bounds for such spaces and gave a development method for transforming local curvature bounds into global ones under certain conditions (see below). We shall use the following terminology. An *interior* metric space $M$ is one in which the distance between any two points is the infimum of the lengths of curves joining them (where curvelength is defined as usual); the terms *inner* and *tight* have also been used. $M$ is a *geodesic space* if in addition it contains a shortest curve between any two points [Gv2]. (*Length space* has been used with both meanings.) A complete interior space is geodesic if it is locally compact [Bu2, p. 24], but might not be otherwise. For instance, an ellipsoid in Hilbert space, the lengths of whose axes are strictly decreasing and bounded above zero, is complete in the interior metric and yet contains infinitely many pairs of points which cannot be joined by shortest curves (see [Gn]). A sequence of intervals of length $1 + 1/n$ with their left and right endpoints respectively identified is complete and locally convex in the interior metric and contains a pair of points not joined by a shortest curve.

A *geodesic* will always be a locally distance-realizing curve parametrized proportionally to arclength by [0, 1]. A geodesic space is *locally convex* if every point has a neighborhood such that the distance $d(\alpha(t), \beta(t))$ is convex for any two minimizing geodesics $\alpha$ and $\beta$ in the neighborhood. (Of course, for Riemannian manifolds without boundary this is equivalent to nonpositive sectional curvature; see [BGS].) If $m_{pq}$ denotes the midpoint of a geodesic from $p$ to $q$, then it is equivalent to say that $M$ is covered by neighborhoods $U$ such that the relation

$$2d(m_{pq}, m_{pr}) \leqslant d(q, r)$$

holds for any three points $p$, $q$ and $r$ in $U$ and any geodesics in $U$ joining them (such geodesics are unique).

We are very grateful to the referee for examining the paper carefully and suggesting a number of technical improvements.

We also thank the referee for informing us of the chapter [Ba] by W. Ballmann that is to appear in *Sur les Groupes Hyperboliques d'après Gromov* (Ghys, de la Harpe, eds.), and its author for promptly sending us a preprint. In [Ba], the Hadamard-Cartan theorem is proved using the Birkhoff curve-shortening technique; this depends on local compactness, which we avoid by exploiting local convexity. Another distinction is that the notions of exponential map and conjugate point are not introduced in [Ba]. The Hadamard-Cartan theorem is applied in [Ba] to obtain a criterion for the hyperbolicity of certain simply connected polyhedra.

## 2. Conjugate points

In a given geodesic space, let $\mathbf{G}_m$ be the space of geodesics starting at $m$, carrying the uniform metric $\mathbf{d}$. Say the point $m$ has *no conjugate points* if the endpoint map on $\mathbf{G}_m$ maps some neighborhood of every $\gamma$ homeomorphically onto a neighborhood of the endpoint of $\gamma$. (In Riemannian manifolds without boundary, this definition is equivalent to the usual one.)

THEOREM 2. *A locally convex, complete geodesic space has no conjugate points.*

Here it is straightforward that the endpoint map is a homeomorphism, and in fact an isometry, from some open neighborhood of every $\gamma$ onto its image. The question is whether it is surjective; that is, whether locally there always exist geodesics from $m$ that vary continuously with their righthand endpoints.