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where the sum runs over all intersections  $U$  of  $i + 1$  distinct elements of  $\mathcal{U}$ . Let  $\check{\delta}: C^i(\mathcal{U}, \mathcal{F}^j) \rightarrow C^{i+1}(\mathcal{U}, \mathcal{F}^j)$  be the Čech co-boundary. We also have boundaries  $d: C^i(\mathcal{U}, \mathcal{F}^j) \rightarrow C^i(\mathcal{U}, \mathcal{F}^{j+1})$ .

Now let

$$C^n(\mathcal{U}, \mathcal{F}^\bullet) = \bigoplus C^p(\mathcal{U}, \mathcal{F}^q)$$

where the sum runs over  $p + q = n$ . For  $c \in C^n(\mathcal{U}, \mathcal{F}^\bullet)$ , we let  $c^{p,q}$  denote its  $p, q$ -th component. The hyper-coboundary

$$\partial: C^n(\mathcal{U}, \mathcal{F}^\bullet) \rightarrow C^{n+1}(\mathcal{U}, \mathcal{F}^\bullet)$$

is defined as follows: For  $c \in C^n(\mathcal{U}, \mathcal{F}^\bullet)$ , we set

$$(\partial c)^{p,q} = dc^{p-1,q} + (-1)^{p-1} \check{\delta} c^{p,q-1} .$$

Then the hypercohomology of  $\mathcal{F}$  with respect to  $\mathcal{C}, \mathbf{H}^\bullet(S, \mathcal{F}^\bullet, \mathcal{C})$ , is defined to be  $\text{Ker}(\partial)/\text{Image}(\partial)$  and  $\mathbf{H}^\bullet(S, \mathcal{F}^\bullet)$  is defined to be an appropriate limit of these groups over all ordered covers. In particular, if  $S$  is a scheme,  $\mathcal{F}^\bullet$  is a complex of coherent sheaves and  $\mathcal{C}$  is an affine open cover, then  $\mathbf{H}^\bullet(S, \mathcal{F}^\bullet)$  is naturally isomorphic to  $\mathbf{H}^\bullet(S, \mathcal{F}^\bullet, \mathcal{C})$ . If in addition  $S$  is affine  $\mathbf{H}^\bullet(S, \mathcal{F}^\bullet) \cong H^\bullet(\Gamma(\mathcal{F}^\bullet))$ .

### 1. EXTENSIONS OF CONNECTIONS

Let  $S$  be smooth connected scheme over a field  $K$  of characteristic zero. Suppose  $(H, \nabla_H)$  and  $(G, \nabla_G)$  are integrable connections on  $S$ . The set of isomorphism classes of integrable extensions of  $(H, \nabla_H)$  by  $(G, \nabla_G)$  forms a group under Baer sum which we will call  $\text{Ext}(H, G)$ .

PROPOSITION 1.1.1.  $\text{Ext}(H, G) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H)$ .

*Proof.* Since  $\nabla_H$  is integrable,  $H$  is locally free. Let  $\mathcal{C}$  be an ordered affine open cover of  $S$  such that  $H(U)$  is a free  $\mathcal{O}_S(U)$ -module for each  $U \in \mathcal{C}$ . Suppose we have an extension

$$0 \rightarrow (G, \nabla_G) \rightarrow (E, \nabla) \rightarrow (H, \nabla_H) \rightarrow 0$$

of connections. Let  $U \in \mathcal{C}$ . Since  $H(U)$  is free, there exists an  $\mathcal{O}_S(U)$ -module section  $s_U: H(U) \rightarrow E(U)$ . Now let  $h_U = \nabla \circ s_U - s_U \circ \nabla_H$ . We claim that  $h_U$  is an  $\mathcal{O}_S(U)$ -module homomorphism from  $H(U)$  into  $\Omega_S^1 \otimes G(U)$ , i.e. an element of  $\text{Hom}_{\mathcal{O}_S}(H, \Omega_S^1 \otimes G)(U)$ . Indeed, for  $f \in \mathcal{O}_S(U)$  and  $v \in H(U)$ ,

$$\begin{aligned} h_U(fv) &= \nabla(s_U(fv)) - s_U(\nabla_H(fv)) = \nabla(fs_U(v)) - s_U(df \otimes v + f\nabla_H v) \\ &= df \otimes s_U(v) - f\nabla(s_U(v)) - (df \otimes s_U(v) + fs_U(\nabla_H v)) = fh_U(v). \end{aligned}$$

Let  $s_{U,V} = s_U - s_V \in \text{Hom}_{\mathcal{O}_S}(H, G)(U \cap V)$ . We claim that  $(\{h_U\}, \{s_{U,V}\})$  is a hyper one-cocycle for the complex  $(\Omega_S^1 \otimes \text{Hom}_{\mathcal{O}_S}(H, G), \nabla_{H,G})$ . First it is clear that  $\{s_{U,V}\}$  is a one-cocycle for the sheaf  $\text{Hom}_{\mathcal{O}_S}(H, G)$ . Second

$$\nabla_G \circ s_{U,V} - s_{U,V} \circ \nabla_H = \nabla \circ (s_U - s_V) - (s_U - s_V) \circ \nabla_H = h_U - h_V.$$

Finally, since

$$\nabla \circ \nabla \circ s_U = \nabla \circ s_U \circ \nabla_H + \nabla \circ h_U = h_U \circ \nabla_H + \nabla_G \circ h_U = \nabla_{H,G}(h_U),$$

(using Lemma 1.0.1)  $\nabla$  is integrable iff  $\nabla_{H,G}(h) = 0$ .

Moreover, suppose  $\{s'_U\}$  is another collection of sections

$$s'_U: H(U) \rightarrow E(U), \quad h'_U = \nabla' \circ s'_U - s'_U \circ \nabla$$

and  $s'_{U,V} = s'_U - s'_V$ . Then  $r_U = s'_U - s_U \in \text{Hom}_{\mathcal{O}_S}(H, G)$  and

$$h'_U = h + \nabla \circ r_U - r_U \circ \nabla_H = h + \nabla_G \circ r_U - r_U \circ \nabla_H = h + \nabla_{H,G}(r_U).$$

And so  $(\{h_U\}, \{s_{U,V}\}) - (\{h'_U\}, \{s'_{U,V}\})$  is the hyper-boundary of  $\{r_U\}$ . Thus we get a natural map from

$$\text{Ext}(H, \mathcal{O}_X) \text{ into } H^1(\text{Hom}_{\mathcal{O}_S}(H, G), \nabla_{H,G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H).$$

It is easy to see that this map is a homomorphism.

We can make a map back as follows. Given a hyper-cocycle  $(\{h_U\}, \{s_{U,V}\})$  for the complex  $(\Omega_S^1 \otimes \text{Hom}_{\mathcal{O}_S}(H, G), \nabla_{H,G})$ , let  $E$  be the sheaf determined by the condition that  $E(U) = G(U) \oplus H(U)$  with gluing data

$$(w, v) \rightarrow (w + s_{U,V}, v)$$

on  $U \cap V$ . We then put a connection  $\nabla$  on  $E$  by setting

$$\nabla(w, v) = (\nabla_G w + h_U(v), \nabla_H v)$$

for local sections  $w$  and  $v$  of  $G$  and  $H$  on  $U$ . One can check easily that  $E$  is an extension of  $H$  by  $G$  and that this construction gives the inverse to the map above.  $\square$

**COROLLARY 1.1.2.**  *$\text{Ext}(H, \mathcal{O}_S)$  is a  $K$  vector space and hence is uniquely divisible.*

**COROLLARY 1.1.3.** *Suppose  $S$  is affine and  $S'$  is a non-empty affine open of  $S$ . Then  $\text{Ext}(H, \mathcal{O}_S)$  injects into  $\text{Ext}(H \otimes \mathcal{O}_{S'}, \mathcal{O}_{S'})$ .*

We note that taking duals yields an isomorphism between  $\text{Ext}(G, H)$  and  $\text{Ext}(\check{H}, \check{G})$ . Also, upon identifying  $(\check{G})^\vee$  with  $G$ ,  $\check{\nabla}_G^\vee = \nabla_G$ .

LEMMA 1.1.4. *The diagram*

$$\begin{array}{ccc} \text{Ext}(H, G) & \rightarrow & H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H) \\ \downarrow & & \downarrow \\ \text{Ext}(\check{G}, \check{H}) & \rightarrow & H^1(\check{H} \otimes G, \check{\nabla}_H \otimes \nabla_G) \end{array}$$

*anti-commutes, where the horizontal arrows are the isomorphisms given by the proposition and the right vertical arrow is the evident one.*

*Proof.* Since the assertion is local, we may suppose  $H$  and  $G$  are free. Suppose  $(E, \nabla)$  is an extension of  $H$  by  $G$  and  $s: H \rightarrow E$  is a section. Then  $h = \nabla \circ s - s \circ \nabla_H$  is an element of  $\text{Hom}_{\mathcal{O}_S}(H, \Omega_S^1 \otimes G)$  which represents the image of the isomorphism class of  $E$  in

$$H^1(\text{Hom}(H, G), \nabla_{H,G}) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H).$$

The image  $k$  of  $h$  in  $\text{Hom}_{\mathcal{O}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$  is determined by

$$k(w)(v) = w(h(v)) = w((\nabla \circ s - s \circ \nabla_H)(v))$$

where  $v$  is a section of  $H$  and  $w$  is a section of  $\check{G}$ .

Now  $(\check{E}, \check{\nabla})$  is an extension of  $\check{G}$  by  $\check{H}$  and the homomorphism  $t$  determined by

$$t(w)(e) = w(e - s \circ \pi(e))$$

is a section, where  $\pi: E \rightarrow H$  is the projection,  $e$  is a section of  $E$  and  $w$  is a section of  $\check{G}$ . Hence,  $g = \check{\nabla} \circ t - t \circ \nabla_G^\vee$  is an element of  $\text{Hom}_{\mathcal{O}_S}(\check{G}, \Omega_S^1 \otimes \check{H})$  which represents the image of the isomorphism class of  $\check{E}$  in

$$H^1(\text{Hom}(\check{G}, \check{H}), \nabla_{\check{G}, \check{H}}).$$

Now

$$g(w)(v) = (\check{\nabla} \circ t - t \circ \nabla_G^\vee)(w)(e)$$

where  $e = s(v)$  and

$$\begin{aligned} \check{\nabla} \circ t(w)(e) &= d(w(e - s(\pi(e)) - w(\nabla(e) - s(\pi(\nabla(e)))) \\ &= -w(\nabla \circ s(v) - s \circ \nabla_H(v)) = -k(w)(v) \end{aligned}$$

since  $\pi(s(v)) = v$  and  $\pi \nabla(e) = \nabla_H(\pi(e))$ . The lemma now follows from

$$(t \circ \nabla_G^\vee)(w)(e) = \nabla_G^\vee(w)(e - s(\pi(e))) = 0. \quad \square$$

Suppose  $W$  is an  $\mathcal{O}_S$  submodule of  $H$ . We let  $[W]$  denote the smallest subconnection of  $H$  containing  $W$ .

## 2. THE GAUSS-MANIN CONNECTION

Here we will recall the definition and some basic properties of the Gauss-Manin connection which we will need in this paper. For more details see [K-O]. If  $\mathcal{S}^\bullet$  is a complex,  $\mathcal{S}^\bullet(k)$  will denote the complex obtained from  $\mathcal{S}^\bullet$  by setting  $\mathcal{S}^i(k) = \mathcal{S}^{i+k}$ . For any scheme  $Y$  over  $K$  will let  $K[Y]$  denote  $\Gamma(\mathcal{O}_Y)$ .

Suppose  $S$  is a smooth connected affine scheme over  $K$ . Suppose  $f: X \rightarrow S$  is a smooth morphism,  $Z$  is a closed subscheme of  $X$ , smooth over  $S$ . Suppose  $T$  is either  $\text{Spec}(K)$  or  $S$ . Then we define the subcomplex  $\Omega_{X/T,Z}^\bullet$  of  $\Omega_{X/T}^\bullet$  by the exactness of the sequence.

$$0 \rightarrow \Omega_{X/T,Z}^\bullet \rightarrow \Omega_{X/T}^\bullet \rightarrow \Omega_{Z/T}^\bullet \rightarrow 0.$$

When  $T = \text{Spec}(K)$  we drop it from the notation. It follows that  $\Omega_{X/S,Z}^i = \Omega_{X/S}^i$  for  $i > \dim_S Z$ . Note that  $\Omega_{X,Z}^0 = \Omega_{X/S,Z}^0$  is the sheaf of ideals of  $Z$  on  $X$ . We define  $H_{DR}^i(X/S, Z)$  to be the  $i$ -th hypercohomology group of the complex  $\Omega_{X/S,Z}^\bullet$ . We set  $H_{DR}^i(X/S) = H_{DR}^i(X/S, \emptyset)$ . If  $X$  is affine, then  $H_{DR}^i(X/S, Z)$  is the  $i$ -th cohomology group of the complex of  $K[S]$  modules  $\Gamma(\Omega_{X/S,Z}^\bullet)$ . If  $X$  is affine,  $K$  has characteristic zero and  $U$  is a dense open subscheme of  $X$  then the natural map from  $H_{DR}^i(X/S, Z)$  to  $H_{DR}^i(U/S, U \cap Z)$  is an injection.

From the last short exact sequence with  $T = S$ , we obtain a long exact sequence

$$(2.1) \quad \dots \rightarrow H_{DR}^{i-1}(Z/S) \rightarrow H_{DR}^i(X/S, Z) \rightarrow H_{DR}^i(X/S) \rightarrow \dots$$

The Gauss-Manin connection  $\nabla: H_{DR}^i(X/S, Z) \rightarrow \Omega_S^1 \otimes H_{DR}^i(X/S, Z)$  is the boundary map in the long exact sequence obtained by taking hypercohomology of the short exact sequence of complexes:

$$(2.2) \quad 0 \rightarrow f^* \Omega_S^1 \otimes \Omega_{X/S,Z}^\bullet(-1) \rightarrow \Omega_{X/S,Z}^\bullet / f^* \Omega_S^2 \otimes \Omega_X^\bullet(-2) \rightarrow \Omega_{X/S,Z}^\bullet \rightarrow 0$$

(which is exact because  $X$  and  $Z$  are smooth over  $S$ ). It is an integrable connection. If  $K$  has characteristic zero and  $f$  is surjective and has geometrically connected fibers, then  $H_{DR}^0(X/S) = K[S]$  and the Gauss-Manin