

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 36 (1990)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** MANIN'S PROOF OF THE MORDELL CONJECTURE OVER  
FUNCTION FIELDS  
**Autor:** Coleman, Robert F.  
**Kapitel:** 4. Abelian schemes  
**DOI:** <https://doi.org/10.5169/seals-57915>

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$$(\{\bar{\omega}_U - d_{X/S}g_U\}, \{f_{U,V} - (g_U - g_V)\})$$

for some one-chain  $\{g_U\}$  with coefficients in  $\mathcal{O}_X$  such that

$$s^*f_{U,V} = u^*f_{U,V} = s^*(g_U - g_V) \quad \text{and} \quad t^*f_{U,V} = v^*f_{U,V} = t^*(g_U - g_V).$$

Let  $\eta_U = \omega_U - dg_U$ . Now

$$s^*\eta_U - s^*\eta_V = s^*df_{U,V} - s^*d(g_U - g_V) = 0$$

by the conditions that  $\{g_U\}$  must satisfy and the fact that  $(\{\omega_U\}, \{f_{U,V}\})$  is a hypercycle. Similarly,  $t^*\eta_U - t^*\eta_V = 0$ . Let  $\eta_s$  and  $\eta_t$  be the elements of  $\Omega_S^1$  determined by the cocycles  $\{s^*\eta_U\}$  and  $\{t^*\eta_U\}$  respectively.

Now to compute  $\nabla h([\omega])$  we must lift  $\bar{\omega}_U - d_{X/S}g_U$  to a section of  $\Omega_{X,Z}^1$ . Let  $e_{s,U}$  and  $e_{t,U}$  be elements of  $\mathcal{O}_X(U)$  such that  $s^*e_{s,U} = 1$ ,  $t^*e_{t,U} = 0$ ,  $t^*e_{s,U} = 0$  and  $s^*e_{t,U} = 1$ . These elements exist since  $Z$  is étale over  $S$ . Then  $\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)$  is such a lifting. To compute  $\nabla h([\omega])$  we must take the hyper-coboundary of  $(\{\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)\}, \{f_{U,V} - (g_U - g_V)\})$ . It is

$$(\{\eta_s \otimes d_{X/S}e_{s,U} + \eta_t \otimes d_{X/S}e_{t,U}\}, \{\eta_s \otimes (e_{s,U} - e_{s,V}) + \eta_t \otimes (e_{t,U} - e_{t,V})\}, 0).$$

The class of this hypercycle is the image of

$$\eta_t - \eta_s \in \Omega_S^1 \quad \text{in} \quad \Omega_S^1 \otimes H_{DR}^1(X/S, Z)$$

(recall that we've determined a map of  $K[S]$  into  $H_{DR}^1(X/S, Z)$ ). Hence  $\nabla h([\omega]) = \eta_t - \eta_s$ .

The proposition now follows from the fact that

$$(\{\eta_s + ds^*g_U\}, \{s^*g_U - s^*g_V\}) = u^*(\{\omega_U\}, \{f_{U,V}\})$$

and

$$(\{\eta_t + dt^*g_U\}, \{t^*g_U - t^*g_V\}) = v^*(\{\omega_U\}, \{f_{U,V}\}). \quad \square$$

**COROLLARY 1.3.4.** *If, in the above,  $u$  and  $v$  are constant, then  $M(s, t) = 0$ .*

#### 4. ABELIAN SCHEMES

Suppose now that  $A$  is an Abelian scheme over  $S$ . Let  $m: A \times_S A \rightarrow A$  be the addition law and  $e$  the zero section. For  $s, t \in A(S)$ , let  $M(s) = M(e, s)$  and  $s + t = m(s, t)$ .

**THEOREM 1.4.1.** *The map  $M$  from  $A(S)$  to  $\text{Ext}(H_{DR}^1(A/S), K[S])$  is a homomorphism.*

*Proof.* Let  $s$  and  $t$  be elements of  $A(S)$ . Define the map  $g: A \rightarrow A$  by  $g = m \circ (id, t \circ f)(g(x) = x + t(f(x)))$ . Then  $g^*: H_{DR}^1(A/S) \rightarrow H_{DR}^1(A/S)$  is the identity so that  $g^*M(e, s) = M(e, s)$  on the one hand and  $g^*M(e, s) = M(t, s + t)$  by Proposition 1.3.2 on the other. Hence,

$$M(s) + M(t) = M(e, s) + M(e, t) = M(t, s + t) + M(e, t) = M(e, s + t)$$

by Proposition 1.3.1.  $\square$

Let  $(B, \tau)$  denote the  $K(S)/K$  trace of  $A_{K(S)}$  (see [L-AV]). In particular,  $B$  is an Abelian scheme over  $K$  and  $\tau: B \times \text{spec}(K(S)) \rightarrow A_{K(S)}$  is a homomorphism. Since  $K$  has characteristic zero  $\tau$  is a closed immersion. Philosophically,  $B$  is the largest constant Abelian subscheme of  $A_{K(S)}$  defined over  $K$ . The morphism  $\tau$  extends uniquely to an  $S$ -morphism  $\bar{\tau}: B \times_K S \rightarrow A$ . It follows that  $B(K)$  maps naturally into  $A(S)$ . We call the elements  $s$  of  $A(S)$  such that  $ns$  is in the image of  $B(K)$ , the constant sections of  $A/S$ .

PROPOSITION 1.4.2. *The kernel of  $M$  contains all constant sections of  $A/S$ .*

*Proof.* Let  $s$  be a constant section of  $A/S$ . Then there exists a positive integer  $n$  such that  $ns = \bar{\tau} \circ (t \times id)$  where  $t \in B(K)$ . Hence it follows from the above theorem, Proposition 1.3.2 and Proposition 1.3.4 that  $nM(s) = M(ns) = M(\bar{\tau}(t \times id)) = \bar{\tau}^*M(t \times id) = 0$ . Since

$$\text{Ext}(H_{DR}^1(A/S), K[S])$$

is uniquely divisible, by Corollary 1.1.2, the proposition follows.  $\square$

We wish to prove the converse of this proposition. I.e. we wish to prove:

THEOREM 1.4.3. *The kernel of  $M$  is precisely the group of all constant sections of  $A/S$ .*

We will give two proofs of this result. The first is Algebraic. The second is analytic and is essentially a reformulation of Manin's proof based on remarks by Katz [K2] in a letter to Ogus.

## 5. THE ALGEBRAIC PROOF

### a. *Differentials with logarithmic singularities*

(See [K] §1.0). Suppose  $X$  is a smooth scheme over a scheme  $T$  and  $Z$  is a hypersurface in  $X$  whose irreducible components are smooth over  $T$  and