

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 36 (1990)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** GAUSS SUMS AND THEIR PRIME FACTORIZATION  
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**Kapitel:** 3. The prime factorization of the Gauss sum: statement of the result  
**DOI:** <https://doi.org/10.5169/seals-57901>

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corresponding to this isomorphism and let  $\mathfrak{P}$  be the prime in  $\mathbf{Q}(pm)$  above  $\mathfrak{p}$ , so  $\mathfrak{P}^{p-1} = \mathfrak{p}$ , if we identify the prime ideal  $\mathfrak{p}$  of  $\mathbf{Q}(m)$  with its extension to a fractional ideal of  $\mathbf{Q}(pm)$ . Thus we have the following congruence

$$(2.1) \quad \chi(x) \equiv x^{(p-1)/m} \pmod{\mathfrak{P}} \quad \text{for all } x \in \mathbf{F}_p^* .$$

Let  $v_{\mathfrak{P}}$  be the valuation on  $\mathbf{Q}(pm)$  corresponding to  $\mathfrak{P}$ . The number  $\zeta_p - 1$  is a uniformizing element of  $v_{\mathfrak{P}}$  in the sense that  $v_{\mathfrak{P}}(\zeta_p - 1) = 1$ . Moreover one has  $v_{\mathfrak{P}}(p) = p - 1$ . From the prime  $\mathfrak{P}$  we get the other primes in  $\mathbf{Q}(pm)$  above  $p$  by Galois action: each prime in  $\mathbf{Q}(pm)$  above  $p$  is equal to  $\mathfrak{P}^{\tau}$ , the image of  $\mathfrak{P}$  under the Galois action of  $\tau$ , for a unique  $\tau \in \text{Gal}(\mathbf{Q}(m)/\mathbf{Q})$ .

(2.2) In the same way we get from the prime  $\mathfrak{p}$  all the primes in  $\mathbf{Q}(m)$  above  $p$ . However, in the last section of this paper, it will be more convenient to use a slightly different description of the primes in  $\mathbf{Q}(m)$  above  $p$ . There we will not fix  $\chi$ , as we do in the rest of the paper, but we will let it run over the  $\phi(m)$  multiplicative characters on  $\mathbf{F}_p$  of order  $m$ . For each such  $\chi$  we let  $\mathfrak{p} = \mathfrak{p}(\chi)$  be the prime in  $\mathbf{Q}(m)$  above  $p$  associated to  $\chi$  in the way described above. Then  $\mathfrak{p} = \mathfrak{p}(\chi)$  runs over the  $\phi(m)$  primes in  $\mathbf{Q}(m)$  above  $p$ .

### 3. THE PRIME FACTORIZATION OF THE GAUSS SUM:

#### STATEMENT OF THE RESULT

Before we state the outcome of the prime factorization of  $G$  we introduce some more notation. For each  $i \in \mathbf{Z}$  with  $0 < i < m$  and  $(i, m) = 1$  we define the integer  $k_i$  to be the exponent of the prime  $\mathfrak{P}^{\tau_i^{-1}}$  in the prime factorization of  $G$  in  $\mathbf{Q}(pm)$  (it turns out that an inverse has to appear somewhere and this is a convenient place). Equivalently,  $k_i$  is the exponent of the prime  $\mathfrak{P}$  in the prime factorization of  $G^{\tau_i}$ , that is,

$$(3.1) \quad k_i = v_{\mathfrak{P}}(G^{\tau_i}) .$$

Any given action of a group  $\Gamma$  on an algebraic number field  $F$  induces an action of the group  $\Gamma$  on  $I(F)$ , the group of fractional ideals in  $F$ . Now we proceed with it just as we did above with the action of  $\Gamma$  on the multiplicative group  $F^*$ : we denote the action of  $\Gamma$  on  $I(F)$  by the

exponential notation, we extend it by  $\mathbf{Z}$ -linearity to an action of the group ring  $\mathbf{Z}\Gamma$  on  $I(F)$  and we denote this action also by the exponential notation. If moreover  $E$  is a subfield of  $F$  then we can view  $I(E)$  as a subgroup of  $I(F)$  by extension of fractional ideals; moreover if  $\alpha \in I(E)$  with  $\alpha = \mathfrak{b}^r$  for some  $\mathfrak{b} \in I(F)$  and some  $r \in \mathbf{N}$  and if  $\lambda \in \mathbf{Q}\Gamma$  with  $r\lambda \in \mathbf{Z}\Gamma$ , then we make as usual the convention that the formal expression  $\alpha^\lambda$  means the fractional ideal  $\mathfrak{b}^{(r\lambda)}$  in  $F$ . We define the Stickelberger element  $\theta$  in the group ring  $\mathbf{Q}[\text{Gal}(\mathbf{Q}(m)/\mathbf{Q})]$  by

$$(3.2) \quad \theta = \sum_i \frac{i}{m} \tau_i^{-1}$$

where  $i$  runs over the positive integers  $< m$  which are relatively prime to  $m$ . The formal expression  $\mathfrak{p}^\theta$  denotes the ideal  $\mathfrak{P}^{(p-1)\theta}$ , by the convention made above for fractional exponents and by the relation  $\mathfrak{p} = \mathfrak{P}^{p-1}$  between  $\mathfrak{p}$  and  $\mathfrak{P}$ .

Now we are ready to formulate the following result of Stickelberger on the Gauss sum  $G$  as defined in (1.1):

(3.3) THEOREM. *The prime factorization of the Gauss sum  $G$  is  $\mathfrak{p}^\theta$ .*

(3.3) The statement of the theorem is clearly equivalent to the following one: only the primes in  $\mathbf{Q}(pm)$  above  $p$  occur in the prime factorization of  $G$ , and their exponents in this factorization are as follows: for each positive integer  $i < m$  which is relatively prime to  $m$ , the exponent of the prime  $\mathfrak{P}^{\tau_i^{-1}}$  is  $k_i = \frac{p-1}{m} i$ .

#### 4. A USEFUL LEMMA

In the proof of theorem (3.3) we will use a simple general lemma to determine the exponents in the prime factorization of the Gauss sum  $G$ . The aim of this section is to state and to prove this lemma. Let  $F$  be a field,  $v$  a discrete valuation on  $F$ ,  $F(v)$  the residue class field of  $v$  and  $\pi$  a uniformizing element of  $v$ , that is,  $\pi \in F^*$  with  $v(\pi) = 1$ . An element  $u \in F^*$  with  $v(u) = 0$  will be called a  $v$ -unit. We define a homomorphism  $l$  from  $F^*$  to  $\mathbf{Z} \times F(v)^*$  by sending each  $\alpha \in F^*$  to the pair  $(k, r)$  consisting of the integer  $k = v(\alpha)$  and the residue class  $r$  in  $F(v)$  of the  $v$ -unit  $\alpha/\pi^k$ . We call  $l(\alpha)$