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THE POMPEIU PROBLEM REVISITED

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ABSTRACT. One of the central results connected with the Pompeiu problem is a theorem of Brown, Schreiber and Taylor. Using some old work of the authors on spectral synthesis, a proof of this result is given. Though separately dealt with, it is shown that some of the main results for the Pompeiu problem for non-Euclidean symmetric spaces can also be treated in the same spirit. In all the cases the role of representations of the underlying group of isometries is highlighted. This point of view leads to some new results for the Pompeiu problem for two sided translations on the non-commutative groups $SL(2, \mathbf{R})$ and $M(2)$. Finally, a brief discussion is provided for some related problems.

1. INTRODUCTION

Let X be a locally compact Hausdorff space and G a group of homeomorphisms of X each of which leaves a given non-negative Radon measure μ invariant. The central theme of this article is what is known in the literature as the Pompeiu property: a relatively compact measurable subset $E \subseteq X$ is said to have the Pompeiu property if for a continuous function f on X,

$$\int_{gE} f(x)d\mu(x) = 0 \quad \text{for all} \quad g \in G \tag{1.1}$$

implies $f \equiv 0$.

The Pompeiu property in a wide variety of settings and its relation to other problems have been the subject-matter of a large number of investigations beginning with two articles by the Roumanian mathematician D. Pompeiu in 1929 ([18], [19]). In the first paper ([18]) the set-up was essentially $X = \mathbf{R}^2$ with the Lebesgue measure μ and G the group $\mathbf{R}^2$

acting through translations, where the unit disc $D \subseteq \mathbf{R}^2$ was claimed to have the Pompeiu property. We need only to look at the Fourier-Laplace transform of the characteristic function 1_D of the disc D to see that if a is a zero of the Bessel function J_1, then $f(x, y) = \sin ax$ is a nonzero function satisfying (1.1). In fact, it was later realised that no bounded subset E has the Pompeiu property, for this set-up (see [12], Theorem 4.3). However, as already seen in the second paper of Pompeiu ([19]), the problem becomes more meaningful, even hard, if either G is replaced by a larger group or further restrictions are imposed on f in condition (1.1).

The basic paper in the theory is, however, the 1973 work of Brown, Schreiber and Taylor ([12]), who considered $X = \mathbf{R}^2$ with Lebesgue measure and $G = M(2)$, the Euclidean motion group (which, they point out, is no different from the more general setting of $X = \mathbf{R}^n$ and $G = M(n)$). They show that the Pompeiu property is closely related to the work of L. Schwartz on mean periodic functions ([23]) — a key observation for their and subsequent work on this theme. Their main result states that E has the Pompeiu property if and only if the Fourier-Laplace transform of the characteristic function 1_E,

$$\hat{1}_E(z_1, z_2) = \int_E e^{-i(t_1 z_1 + t_2 z_2)} dt_1 dt_2, \ (z_1, z_2) \in \mathbf{C}^2 \tag{1.2}$$

does not vanish identically on any of the varieties

$$M_\alpha = \{(z_1, z_2) \in \mathbf{C}^2 : z_1^2 + z_2^2 = \alpha^2\}, \quad 0 \neq \alpha \in \mathbf{C}.$$

By direct computation, these authors are able to verify condition (1.2) for all proper ellipses: thus $E = \{(x, y) : x^2/a^2 + y^2/b^2 \leqslant 1\}$ with $ab \neq 0$, $a \neq b$ has the Pompeiu property. Such direct computation can work only for sets with rigid geometric properties. Brown, Schreiber and Taylor have a general result: a bounded simply connected domain whose boundary has a "corner" has the Pompeiu property ([12], Theorem 5.11; see Section 3 for a precise statement). Triangles, parallelograms and polygonal figures thus have the Pompeiu property.

Another formulation of the problem (see [3], [33], [34]) is that, if E is a *simply connected bounded domain with a Lipschitz boundary*, then condition (1.2) holds if and only if there is a complex number $\alpha \neq 0$ such that the over-determined boundary value problem

$$\begin{aligned} \Delta T + \alpha T &= 0 \quad \text{on} \quad E \\ T = \text{constant} \neq 0 \text{ on } \partial E, \partial T/\partial n &\equiv 0 \quad \text{on} \quad \partial E \end{aligned} \tag{1.3}$$

has a solution. S. A. Williams ([34]) used (1.3) to show that if E fails to have the Pompeiu property then ∂E is real-analytic. In yet another development C. A. Berenstein showed that if E is a simply-connected bounded domain with smooth boundary and $\hat{1}_E$ vanishes on an infinite sequence of varieties $M_{\alpha_1}, M_{\alpha_2}, \ldots$, then indeed E is a disc (see [3]).

In the more general setting of a Riemannian symmetric space with the associated Riemannian volume measure and G a group of isometries, the problem has been studied in depth by Berenstein and Zalcman ([9]) and Berenstein and Shahshahani ([7]). Here again, the question reduces to one of spectral analysis in Euclidean spaces. Since Schwartz's theorem holds in dimension one only, definitive results could be obtained for rank-1 symmetric spaces alone. The differential equation approach has been fruitful in this case also.

In this paper we present a brief survey of the development outlined above. Apart from the original paper of Brown, Schreiber and Taylor ([12]), a proof of their main theorem can also be found in Berenstein and Zalcman [9] as a particular case of their more general set-up. In this paper we give a proof of the main theorem of Brown, Schreiber and Taylor and its analogue for non-compact symmetric spaces in line with our approach to the problem of spectral analysis in [1]. This, we believe, has the merit of being more transparent, at least for the somewhat less general form of the theorem that we consider here. Our proof also provides an application of the main results in [1]. We are also able to treat some analogous results of Berenstein and Zalcman ([9]) for symmetric spaces of the compact type in the same spirit. We then consider the Pompeiu problem in the context of the group $SL(2, \mathbf{R})$ and $M(2)$ and derive some results based on their representation theory.

The paper is organised as follows. Instead of trying to unify the treatment of the problem for the symmetric spaces of the three types (compact, non-compact and Euclidean) we choose to present them separately: the spaces of Euclidean type in Section 3, those of non-compact type in Section 5 and the compact case in Section 6. We take care, however, to stress the basic similarity and to bring out the role of the so-called class-1 representations in each case. In Section 2, we discuss spectral analysis of radial functions on $\mathbf{R}^n$ — for use in Section 3. In Section 4, we discuss a conjecture which remains a major open question in the theory. In Section 7, we consider what can be called the Pompeiu problem on groups — an area that remains largely unexplored yet. Finally, in Section 8, we try to provide a brief survey of the literature on some allied problems. Though extensive, our bibliography is far from complete. We refer the reader to the bibliographies in [9], [12] and [35].

2. Spectral analysis of radial functions

We denote by $\mathscr{E}(\mathbf{R}^n)$, the space of C^∞ functions on $\mathbf{R}^n$ with the usual topology and by $\mathscr{E}'(\mathbf{R}^n)$, the dual space of distributions of compact support with the strong topology — both Fréchet-Montel and hence reflexive spaces. $C_c^\infty(\mathbf{R}^n)$ is the space of C^∞-functions of compact support. For a space of functions or distributions $\mathscr{F}$, we denote the usual action of an element σ of the orthogonal group $O(n, \mathbf{R})$ by the notation $f \to f^\sigma$. $\mathscr{F}_{\text{rad}}$ will stand for the space of those $f \in \mathscr{F}$ which are invariant under $O(n, \mathbf{R})$, i.e., $f^\sigma = f$ for all $\sigma \in O(n, \mathbf{R})$. $\mathscr{E}'(\mathbf{R}^n)_{\text{rad}}$ is a closed subspace of $\mathscr{E}'(\mathbf{R}^n)$ and the spaces $\mathscr{E}(\mathbf{R}^n)_{\text{rad}}$ and $\mathscr{E}'(\mathbf{R}^n)_{\text{rad}}$ are (strong) duals of each other. In the case $n = 1$, even functions are the analogues of radial functions and we write $\mathscr{F}_e$ to mean $\mathscr{F}_{\text{rad}}$. Though our considerations in this section hold for all $n \geqslant 2$, we shall restrict ourselves to the case $n = 2$ to keep the exposition simple.

We start with a slightly weaker version of the classical theorem of L. Schwartz ([23]).

THEOREM 2.1 (L. Schwartz's theorem on spectral analysis). *Let $\mathscr{U}$ be a nontrivial closed subspace of $\mathscr{E}(\mathbf{R})$, which is closed under translations, then $\mathscr{U}$ contains an exponential function $e^{i\lambda x}$ for some $\lambda \in \mathbf{C}$.*

As pointed out in [1] an immediate corollary of the theorem is: If $\mathscr{U}$ is a nontrivial closed subspace of $\mathscr{E}(\mathbf{R})_e$ which is closed under convolution against all $T \in \mathscr{E}'(\mathbf{R})_e$, then $\mathscr{U}$ contains a function of the form $\psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$.

We now introduce a family of functions on $\mathbf{R}^2$ which is central to spectral analysis of radial functions. For $\lambda \in \mathbf{C}$, define

$$\phi_\lambda(x) = \int_{|w|=1} e^{-i\lambda(x \cdot w)} dw, \; x \in \mathbf{R}^2$$

where the integral is with respect to the normalised Lebesgue measure on the unit circle. Here $x \cdot w$ is the usual inner product. It is immediate that ϕ_λ is a radial function for each $\lambda \in \mathbf{C}$. For $f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}}$, we define a transform (sometimes called the Bessel transform):

$$\mathcal{G}f(\lambda) = \int_{\mathbf{R}^2} \phi_\lambda(x) f(x) dx\,, \quad \lambda \in \mathbf{C}\,.$$

We see that if $\hat{f}$ is the Fourier-Laplace transform of $f \in C_c^\infty(\mathbf{R}^2)$, i.e.,

$$\hat{f}(z_1, z_2) = \int_{\mathbf{R}^2} e^{-i(z \cdot x)} f(x) dx\,, \quad z = (z_1, z_2) \in \mathbf{C}^2\,,$$

then we have

$$\mathcal{G}f(\lambda) = \hat{f}(\lambda, 0)\,, \ \lambda \in \mathbf{C}\,, \ f \in C_c^\infty(\mathbf{R}^2)_{\mathrm{rad}}\,. \tag{2.1}$$

Both the transforms $\mathcal{G}$ defined above and the Fourier-Laplace transform have their obvious extension to $\mathcal{E}'(\mathbf{R}^2)_{\mathrm{rad}}$. We have for $T \in \mathcal{E}'(\mathbf{R}^2)_{\mathrm{rad}}$,

$$\mathcal{G}T(\lambda) = T(\phi_\lambda)\,, \quad \lambda \in \mathbf{C}$$

$$\hat{T}(z_1, z_2) = T(e_z)\,, \quad z \in \mathbf{C}^2$$

where $e_z(x) = e^{-i(z_1 x_1 + z_2 x_2)} = e^{-i(z \cdot x)}$. We again have

$$\mathcal{G}T(\lambda) = \hat{T}(\lambda, 0)\,, \quad \lambda \in \mathbf{C}\,.$$

By applying the Paley-Wiener theorem we are able to obtain a description of the function space $\mathcal{X} = \{\mathcal{G}T : T \in C_c^\infty(\mathbf{R}^2)_{\mathrm{rad}}\}$.

LEMMA 2.2. *$\mathcal{X}$ is the space of even entire functions f on $\mathbf{C}$ such that for some constants, c, N and A (depending on f),*

$$|f(\lambda)| \leqslant C(1+|\lambda|)^N e^{A|\mathrm{Im}\,\lambda|}\,, \quad \lambda \in \mathbf{C}\,.$$

Proof. By the Paley-Wiener theorem an entire function $\phi = \hat{T}$ for some $T \in \mathcal{E}'(\mathbf{R}^2)_{\mathrm{rad}}$ if and only if for some $C, N, A > 0$,

$$|\phi(z)| \leqslant C(1+|z|)^N e^{A|\mathrm{Im}\,z|}$$

and

$$\phi(z) = \phi(\sigma z)$$

for all $z = (z_1, z_2) \in \mathbf{C}^2$ and $\sigma \in SO(2, \mathbf{R})$ (here $\mathrm{Im}\,z = (\mathrm{Im}\,z_1, \mathrm{Im}\,z_2)$. The latter condition is equivalent to saying $\phi(z) = \phi(z')$ whenever $z_1^2 + z_2^2 = z_1'^2 + z_2'^2$. To see this, consider, for each $\alpha \in \mathbf{C}$,

$$M_\alpha = \{z : z_1^2 + z_2^2 = \alpha^2\}\,.$$

If $\alpha \neq 0$, M_α is a connected analytic submanifold of $\mathbf{C}^2$ of complex dimension 1 and $SO(2, \mathbf{R})\,(\alpha, 0, ..., 0)$ is a real submanifold of M_α of dimension 1 on which the analytic function ϕ is given to be constant. This forces ϕ to be a constant on M_α. A modification of the argument is necessary for $\alpha = 0$.

The lemma now follows from the simple observation that if $\lambda^2 = z_1^2 + z_2^2$, then $(\operatorname{Im} \lambda)^2 \leqslant (\operatorname{Im} z_1)^2 + (\operatorname{Im} z_2)^2$ and from the relation 2.1.

A straight-forward application of the one-dimensional Paley-Wiener theorem for even distributions of compact support will show that $\mathscr{X}$ is also equal to

$$\{\hat{T} : T \in \mathscr{E}'(\mathbf{R})_e\}\,.$$

This identification allows us to define the linear map Σ by

$$\Sigma : \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}} \to \mathscr{E}'(\mathbf{R}^2)_e$$
$$(\Sigma T)\hat{}\,(\lambda) = \mathscr{G}T(\lambda)\,, \quad \lambda \in \mathbf{C}\,.$$

Σ is one-to-one and onto. Moreover, we have the following description of the strong topology in $\mathscr{E}'(\mathbf{R}^n)$ (see [12], prop. 2.1): $T_n \to T$ if and only if (i) $\hat{T}_n \to \hat{T}$ uniformly on compact sets along with the derivatives and (ii) $\hat{T}_n$, $n \geqslant 1$ satisfy the uniform Paley-Wiener condition:

$$|\,\hat{T}_n(z)\,| \leqslant C(1+|z|)^N e^{A|\operatorname{Im} z|}\,, \quad z \in \mathbf{C}^n\,.$$

for some $C, N, A > 0$. This description coupled with the observation in the last step of the proof of Lemma 2.2 gives that Σ is a topological linear isomorphism between $\mathscr{E}'(\mathbf{R}^2)_{rad}$ and $\mathscr{E}'(\mathbf{R})_e$ preserving convolution.

On using the reflexivity of $\mathscr{E}(\mathbf{R})_e$ and $\mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}}$ we now get the map $\tilde{\Sigma}$:

$$\tilde{\Sigma} : \mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}} \to \mathscr{E}(\mathbf{R})_e\,,$$
$$<\Sigma(T), \tilde{\Sigma}(f)> \; = \; <T, f> \quad T \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}},\, f \in \mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}}$$

where $< \cdot , \cdot >$ is the pairing of dual spaces.

We now have the following useful lemma:

LEMMA 2.3. *With the notation above, we have*

$$\Sigma(T) * \tilde{\Sigma}(f) = \tilde{\Sigma}(T * f)$$

for all $T \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}$ *and* $f \in \mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}}$, *where* $*$ *denotes the usual convolution on* $\mathbf{R}^2$.

Proof. Let $S \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}$. Consider

$$<\Sigma(S), (\Sigma(T) * \tilde{\Sigma}(f))>

$$= \Sigma(S) * \big(\Sigma(T) * \tilde{\Sigma}(f)\big)^{\vee}(0)$$
$$\text{(where } g^{\vee}(x) = g(-x), g \in \mathscr{E}(\mathbf{R}^2), x \in \mathbf{R}^2\text{)},$$
$$= \Sigma(S) * \big((\tilde{\Sigma} f)^{\vee} * \Sigma(T)^{\vee}\big)(0)$$
$$= \Sigma(S) * \big(\tilde{\Sigma}(f) * \Sigma(T)\big) \text{ as } \Sigma(T), \tilde{\Sigma}(f) \text{ are even}$$
$$= \Sigma(S) * \big(\Sigma(T) * \tilde{\Sigma} f\big)(0)$$
$$= <\Sigma(S*T), \tilde{\Sigma} f>$$
$$\text{(using } \Sigma S * \Sigma T = \Sigma S * T\text{)}$$
$$= <S*T, f>$$
$$= S * T * f(0) \text{ as} f \text{ is even}$$
$$= <S, T*f> .$$

On the other hand,

$$<\Sigma(S), \tilde{\Sigma}(T*f)> = <S, T*f> .$$

The lemma is proved.

Finally, we come to the main result of the section: the spectral analysis theorem for radial functions. As we remarked in the introduction, the development in this section is along the same lines as in [1] where the corresponding result for rank-1 non-compact symmetric spaces is proved.

THEOREM 2.4. *Let* $\mathscr{V}$ *be a closed nonzero subspace of* $\mathscr{E}(\mathbf{R}^2)_{\text{rad}}$ *such that for all* $T \in \mathscr{E}'(\mathbf{R}^2)_{\text{rad}}$ *and* $f \in \mathscr{V}, T * f \in \mathscr{V}$. *Then there exists* $\lambda \in \mathbf{C}$ *such that* $\phi_\lambda \in \mathscr{V}$.

Proof. Consider the closed and nontrivial subspace M of $\mathscr{E}(\mathbf{R})_e$ such that $\tilde{\Sigma}(\mathscr{V}) = M$. By Lemma 2.3, M is closed under convolution with elements $S \in \mathscr{E}'(\mathbf{R})_e$. By the remarks following Theorem 2.1 now, there exists $\lambda \in \mathbf{C}$ such that the functions $\Psi_\lambda \in M$, where $\Psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$, $x \in \mathbf{R}$. A simple calculation now shows

$$<\phi_\lambda, f> = <\Psi_\lambda, \Sigma f> \quad f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}} \subseteq \mathscr{E}'(\mathbf{R}^2)_{\text{rad}} .$$

Thus $\tilde{\Sigma}\phi_\lambda = \Psi_\lambda$ and hence $\phi_\lambda \in \mathscr{V}$.

3. POMPEIU PROBLEM FOR THE $M(2)$ ACTION ON $\mathbf{R}^2$

The Euclidean motion group $M(2)$ is the semidirect product of $\mathbf{R}^2$ with the rotation group $SO(2, \mathbf{R})$.

$$M(2) = \{(x, \sigma): x \in \mathbf{R}^2, \sigma \in SO(2, \mathbf{R})\}$$

where

$$(x, \sigma) \cdot (x', \sigma') = (x + \sigma x', \sigma\sigma')$$

is the group multiplication and an element (x, σ) acts on $y \in \mathbf{R}^2$ by the rule $(x, \sigma)y = \sigma y + x$.

Let E be a relatively compact subset of $\mathbf{R}^2$ of positive Lebesgue measure. If $f \in C(\mathbf{R}^2)$, the space of continuous functions on $\mathbf{R}^2$, the vanishing of the integrals

$$\int_{gE} f(x)dx = 0, \quad \textit{for all} \quad g \in M(2)$$

$$\text{i.e.} \quad \int_{\sigma E + y} f(x)dx = 0, \quad \textit{for all} \quad \sigma \in SO(2, \mathbf{R}), \quad y \in \mathbf{R}^2$$

can be restated as $f * \check{1}_{\sigma E} \equiv 0$, *for all* $\sigma \in SO(2, \mathbf{R})$ or, equivalently $f^\sigma * \check{1}_E \equiv 0$ for all $\sigma \in SO(2, \mathbf{R})$, where $f^\sigma(x) = f(\sigma x)$ and $\check{1}_E(x) = 1_E(-x)$, $x \in \mathbf{R}^2$. We write

$$\mathscr{U} = \{f \in \mathscr{E}(\mathbf{R}^2): f^\sigma * \check{1}_E = 0 \quad \textit{for all} \quad \sigma \in SO(2, \mathbf{R})\}.$$

From elementary smoothing arguments, it follows that E has the Pompeiu property if and only if $\mathscr{U} = \{0\}$. $\mathscr{U}$ is a closed subspace of $\mathscr{E}(\mathbf{R}^2)$ which is invariant under translation and rotation. Let again

$$\mathscr{V} = \{f \in \mathscr{E}(\mathbf{R}^2)_{\text{rad}}: f * \check{1}_E = 0\}.$$

Then $\mathscr{V} \subseteq \mathscr{U}$, $\mathscr{V}$ is a closed subspace of $\mathscr{E}(\mathbf{R}^2)_{\text{rad}}$ and $T * \mathscr{V} \subseteq \mathscr{V}$ for all $T \in \mathscr{E}'(\mathbf{R}^2)_{\text{rad}}$.

We now prove the main theorem of [12] mentioned in the Introduction. (However, we restrict ourselves to indicator functions of sets, rather than general distributions of compact support.)

THEOREM 3.1 (Brown, Schreiber and Taylor). *A relatively compact subset* $E \subseteq \mathbf{R}^2$ *of positive Lebesgue measure does* not *have the Pompeiu property if and only if there exists* $\alpha \in \mathbf{C}$, $\alpha \neq 0$ *such that*

$$\hat{1}_E(z_1, z_2) = 0 \quad \textit{whenever} \quad z_1^2 + z_2^2 = \alpha^2,$$

where $\hat{1}_E$ *is the Laplace-Fourier transform of the characteristic function* 1_E *of* E.

Proof. The *if* part is immediate; for instance, take any $z = (z_1, z_2)$ such that $z_1^2 + z_2^2 = \alpha^2$ and consider the function $e^{iz \cdot x}$. To prove the *only if* part, suppose E has the Pompeiu property. Let $\mathcal{U}$ and $\mathcal{V}$ be defined as above; by assumption we have $\mathcal{U} \neq \{0\}$. We shall now prove that $\mathcal{V} \neq \{0\}$. Choose $f \in \mathcal{U}$ with $f(0) \neq 0$ (this is possible as $\mathcal{U}$ is translation-invariant). Define

$$h(y) = \int_{SO(2,\mathbf{R})} f(\sigma y) d\sigma, \quad y \in \mathbf{R}^2.$$

As $\mathcal{U}$ is $SO(2, \mathbf{R})$-invariant, the function $h \in \mathcal{U}$. Further, h is a radial function by definition, so $h \in \mathcal{V}$. But then $h(0) = f(0) \neq 0$. Thus $\mathcal{V} \neq \{0\}$ and by Theorem 2.2, we have $\lambda \in \mathbf{C}$ such that $\phi_\lambda \in \mathcal{V}$. Further, ϕ_0 is the constant function and hence ϕ_0 cannot belong to $\mathcal{V} \subseteq \mathcal{U}$. So $\lambda \neq 0$ and $\phi_\lambda \in \mathcal{V}$ and, in particular, $\phi_\lambda * \check{1}_E(0) = 0$. In the notation of Section 2, this means $\mathcal{G}1_E(\lambda) = 0$ and hence $\hat{1}_E(\lambda, 0) = 0$. The $SO(2, \mathbf{R})$-invariance of $\mathcal{U}$ now shows that $\hat{1}_E$ vanishes on $SO(2, \mathbf{R}).(\lambda, 0)$. The analyticity argument in Lemma 2.2 will now prove that $\hat{1}_E$ vanishes at all (z_1, z_2) where $z_1^2 + z_2^2 = \lambda^2$. This proves the theorem.

The condition in Theorem 3.1 can also be given a representation theoretic interpretation in terms of the so-called class-1 principal series representation of $M(2)$ — see Section 7.3 for a more precise statement. As we remarked earlier, the condition of the theorem is verifiable only for sets having strong geometric properties. We quote two results from [12] without proof.

THEOREM 3.2 (Brown, Schreiber and Taylor). *The ellipse*

$$E = \left\{(x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\right\}$$

has the Pompeiu property if and only if $a, b > 0$ *and* $a \neq b$.

When $a = b > 0$, D is the disc and we have

$$\hat{1}_D(z_1, z_2) = \text{const. } J_1(\sqrt{z_1^2 + z_2^2}) \,/\, \sqrt{z_1^2 + z_2^2}, \quad (z_1, z_2) \in \mathbf{C}^2,$$

where J_1 is the Bessel function. Since there are infinitely many zeros of J_1, D does not have the Pompeiu property. The next theorem, obtained through a careful estimate ([12]) needs a definition.

Definition 3.3. Let $\Gamma = \Gamma(t)$, $-1 \leqslant t \leqslant 1$ be a Lipschitz curve in $\mathbf{R}^2$ with well defined (a.e.) unit tangent vectors $T(t) = \Gamma'(t) / |\Gamma'(t)|$. The point $p = \Gamma(0)$ is a corner of Γ if both the right and the left limits of $T(t)$ as $t \to 0$ exist and are not multiples of each other.

THEOREM 3.4 (Brown, Schreiber and Taylor). *Let* Ω *be a compact connected subset of* $\mathbf{R}^2$. *Suppose that there is a half-plane* H *and a unique point* $p \in \Omega \cap H$ *of maximal distance from the boundary* ∂H *of* H. *If the boundary of* Ω *near* p *is given by a Lipschitz curve with a corner at* p *then* Ω *has the Pompeiu property.*

Let now Ω be a bounded Borel subset of the plane of positive measure and suppose that Ω does not have the Pompeiu property. By Theorem 3.1, $\hat{1}_\Omega$ vanishes on the algebraic variety $M_\alpha = \{(z_1, z_2): z_1^2 + z_2^2 = \alpha^2\}$ for some $\alpha \neq 0$. As observed in [33] and [34], $\hat{1}_\Omega/(z_1^2+z_2^2-\alpha^2)$ is now an entire function and standard Paley-Wiener theorem yields the following proposition.

PROPOSITION 3.5. *If* Ω *is a bounded Borel subset of* $\mathbf{R}^2$ *of positive measure with* $\hat{1}_\Omega$ *vanishing on* $M_\alpha, \alpha \neq 0$, *then the function* $g(z_1, z_2) = \hat{1}_\Omega/(z_1^2 + z_2^2 - \alpha^2)$ *is an entire function on* $\mathbf{C}^2$ *which is the Laplace-Fourier transform of a distribution of compact support.*

Proposition 3.5 immediately gives rise to a partial differential equation. For, if T is the distribution whose Fourier transform is g, then from

$$(z_1^2+z_2^2-\alpha^2)g(z_1, z_2) = \hat{1}_\Omega(z_1, z_2)$$

we have

$$\Delta T + \alpha^2 T = -1_\Omega \tag{3.1}$$

where Δ is the Laplacian. Conversely, if there exists a distribution T of compact support satisfying the equation (3.1), then $\hat{1}_\Omega$ vanishes on M_α and hence Ω does not have the Pompeiu property. We also remark that if Ω is, further, a bounded simply connected open set and the equation (3.1) has a solution, then α^2 is necessarily a positive real number as can be seen from a simple Green's theorem argument (see [34] for a proof). The equation has been studied in [3], [33] and [34]. In [33] it was proved that a solution of (3.1), if it exists is actually a function. We shall discuss some more of these results in the next section. We end the present section by quoting the main theorem of [34]. This result extends Theorem 3.4 and, barring sets of rotational symmetry all known sets failing to have the Pompeiu property are covered by this result. For a bounded subset $\Omega \subseteq \mathbf{R}^2$, we denote by $\partial^*\Omega$ the boundary of the unbounded component.

THEOREM 3.5 (S. A. Williams). *Let* Ω *be a bounded open subset such that the equation* $\Delta T + \alpha^2 T = -1_\Omega$ *has a function solution of compact support for some* $\alpha > 0$. *Let* R, K, L *be positive real numbers such that* $L > KR$. *Assume that for* $P \in \partial^*\Omega$ *there exists a coordinate system* (x, y) *around* P *so that*

(i) $Q = (-R, R) \times (-L, L)$ *intersects* $\partial\Omega$ *in the graph* $y = f(x)$ *of a Lipschitz function* f *with Lipschitz constant* K, *and*

(ii) $Q \cap \Omega = \{(x, y) : |x| < R$ *and* $f(x) < y < L\}$.

Then f *is real analytic in a neighbourhood of* P.

Thus if we restrict ourselves to the class $\mathscr{D}$ of simply connected bounded open sets with Lipschitz boundary then $\Omega \in \mathscr{D}$ can fail to have the Pompeiu property only if $\partial\Omega$ is real analytic.

4. A LONG-STANDING CONJECTURE !

The following Conjecture has received quite some attention in the literature ([3], [10], [34]).

Conjecture. If $\Omega \subseteq \mathbf{R}^2$ is in the class $\mathscr{D}$ described above and if Ω does not have the Pompeiu property, then Ω is a disc.

As pointed out before, the work of Williams shows that is is enough to consider Ω with $\partial\Omega$ real analytic. For $\Omega \in \mathscr{D}$, the existence of (a necessarily positive) α^2 for which (3.1) has a distribution solution of compact support is equivalent to the existence of a positive γ for which the following overdetermined system has a solution.

(4.1)
$$\Delta T + \gamma T = 0 \quad \text{on} \quad \Omega$$
$$T = \text{constant} \neq 0 \quad \text{on} \quad \partial\Omega, \partial T/\partial n \equiv 0 \quad \text{on} \quad \partial\Omega$$

(see [34] for details). Thus the conjecture can be stated as follows:

If for $\Omega \in \mathscr{D}$, there exists $\gamma > 0$ for which (4.1) admits a solution, then Ω is a disc.

It is remarked in [34] that the conjecture is closely related to a result of Serrin ([25]): If Ω is a bounded connected open set with smooth boundary on which

$$\Delta u = -1 \quad \text{on} \quad \Omega$$
$$u = 0, \partial u/\partial n = \text{constant} \quad \text{on} \quad \partial\Omega$$

has a function solution, then Ω must be a disc.

We now state two partial answers to the conjecture that seem to support the conjecture.

THEOREM 4.1 (Berenstein [3]). *Let* Ω *be a simply connected bounded open subset of* $\mathbf{R}^2$ *with* $C^{2+\varepsilon}$ *boundary, where* $\varepsilon > 0$. *Assume that the boundary value problem (4.1) has solutions for infinitely many positive* γ, *then* Ω *is a disc.*

We need some notation for the next result due to Brown and Kahane ([10]). Let Ω be a convex bounded open connected subset of $\mathbf{R}^2$. For $0 \leqslant \theta < \pi$, let $\omega(\theta)$ be the distance between the two parallel support lines for Ω which make an angle θ with the positive real axis. We assume ∂D is smooth so that ω is a continuous function. Let

$$m(\Omega) = \inf\{\omega(\theta): 0 \leqslant \theta < \pi\} \quad \text{and} \quad M(\Omega) = \sup\{\omega(\theta): 0 \leqslant \theta < \pi\}.$$

THEOREM 4.2 (Brown and Kahane [10]). *Let* Ω *be a convex region of* $\mathbf{R}^2$ *with* $\partial\Omega$ *real analytic. If* $m(\Omega) \leqslant \frac{1}{2} M(\Omega)$, *then* Ω *has the Pompeiu property.*

We remark that the proof of this Theorem is elementary and very elegant.

5. POMPEIU PROPERTY IN NON-COMPACT SYMMETRIC SPACES

Let G *be a connected non-compact semisimple Lie group having finite centre and real rank 1.* Let K be a fixed maximal compact subgroup of G. The space G/K is then a globally symmetric space of the non-compact type of rank 1. G/K is equipped with a natural Riemannian structure with respect to which G acts as a group of isometries and the associated Riemannian volume element μ is G-invariant. The basic results for the Pompeiu problem in this set-up are due to Berenstein and Zalcman ([9], [4]) and Berenstein and Shahshahani ([7]). In [9], the Fourier-analytic characterisation of a set — in fact, more generally, a collection of sets — having the pompeiu property is obtained and some explicit computations are made for geodesic spheres. In [7], the Pompeiu problem is reduced to an eigenvalue problem as in Section 4 and the analogue of Williams's results is obtained. We shall mainly present here a result implicit in the work of Berenstein and Zalcman as well as Berenstein and Shahshahani from our point of view of spectral analysis developed in [1]. It is to be noted that the close connection between the Pompeiu problem and spectral analysis is also developed in [4]. Our treatment, however, is different in that it relies on the results of spectral analysis in [1].

Let $G = KAN$ be the Iwasawa decomposition of G. Let $\mathscr{G}$ be the Lie algebra of G, $\mathscr{A}$ the Lie subalgebra of $\mathscr{G}$ corresponding to A. Since G has rank 1, $\mathscr{A}$ is 1-dimensional. Using the linear functional ρ on $\mathscr{A}$, which is the half-sum of the positive roots for the adjoint action of $\mathscr{A}$ on $\mathscr{G}$, we write the real dual $\mathscr{A}^*$ of $\mathscr{A}$ as $\mathscr{A}^* = \{t\rho, t \in \mathbf{R}\}$ and its complexification $\mathscr{A}_c^* = \{\lambda\rho, \lambda \in \mathbf{C}\}$. For $\lambda \in \mathbf{C}$, we denote by ϕ_λ the elementary spherical function associated with $\lambda\rho \in \mathscr{A}_c^*$. (These functions essentially parametrize the so-called class-1 representations of G.) The Weyl group in this case is the group of order 2 generated by the reflection $\lambda\rho \to -\lambda\rho$ and we have $\phi_{\lambda'} = \phi_\lambda$ if and only if $\lambda = \lambda'$ or $\lambda = -\lambda'$.

For $\lambda \in \mathbf{C}$ and k a non-negative integer, we write

$$\phi_{\lambda,k}(x) = d^k/d\lambda^k\, \phi_\lambda(x), \quad x \in G;$$

in particular, $\phi_{\lambda,0} = \phi_\lambda$. The functions $\{\phi_{\lambda,k}\}$ are K-bi-invariant, i.e.

$$\phi_{\lambda,k}(\kappa g \kappa') = \phi_{\lambda,k}(g), \quad \kappa \in K, \quad g \in G.$$

We denote by $\mathscr{E} = C^\infty(K\backslash G/K)$ the space of all K-bi-invariant C^∞-functions on G, with the topology of uniform convergence on compacta along with "derivatives". By $\mathscr{E}'$ we denote the dual space of $\mathscr{E}$, the space of K-bi-invariant distributions on G of compact support. A closed subspace $\mathscr{U} \subseteq \mathscr{E}$ is called a variety if $T * f \in \mathscr{U}$, whenever $T \in \mathscr{E}'$ and $f \in \mathscr{U}$ (here $*$ denotes convolution in G). The main theorem of [1] can be stated as follows:

THEOREM 5.1. *Let $\mathscr{U} \subseteq \mathscr{E} = C^\infty(K\backslash G/K)$ be a variety. Then $\mathscr{U}$ is the closed linear span in $\mathscr{E}$ of the subset $\{\phi_{\lambda,k}: \lambda \in \mathbf{C}, k \geqslant 0, \phi_{\lambda,k} \in \mathscr{U}\}$. In particular, if $\mathscr{U}$ is nonzero, then there exists $\lambda \in \mathbf{C}$ such that $\phi_\lambda \in \mathscr{U}$.*

We point out that the main ingredients of the proof of Theorem 5.1 are Schwartz's theorem on mean periodic functions on $\mathbf{R}$, and the topological isomorphism of $C_c^\infty(K\backslash G/K)$ with a space of entire functions through the spherical Fourier transform

$$f \to \hat{f}(\lambda) = \int_G f(x)\phi_\lambda(x^{-1})dx, \quad f \in C_c^\infty(K\backslash G/K).$$

The details of this topological isomorphism are also available in [4].

The spherical functions $\{\phi_\lambda : \lambda \in \mathbf{C}\}$ are intimately related to the *class-1 principal series representations* $\{\Pi_\lambda : \lambda \in \mathbf{C}\}$ of G. These representations are all realised in the space $L^2(K/M)$ (with normalised Haar measure dk; here, M is the centraliser of A in K). In fact, for $g \in G$ the operator $\Pi_\lambda(g)$ is:

$$\Pi_\lambda(g)(F)(k) = e^{(i\lambda - \rho)H(g^{-1}k)} F(\kappa(g^{-1}k)), \quad F \in L^2(K/M), \quad k \in K$$

where for any $y \in G$, $y = \kappa(y) \exp H(y) n(y)$ is the Iwasawa decomposition of y with exp denoting the exponential map: $\mathscr{A} \to A$. Then, $\phi_\lambda(g) = \langle \Pi_\lambda(g)1, 1 \rangle$, $g \in G$, where 1 is the constant function in $L^2(K/M)$ and $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(K/M)$.

For a function f on $X = G/K$, we shall denote by $\tilde{f}$ the right K-invariant lift of f on G: $\tilde{f}(g) = f(gK)$. Similarly, if $E \subseteq X$, we write $\tilde{E} = \{g \in G : gK \in E\}$. As before, if f is a function on G, we denote by $\check{f}$ the function

$$\check{f}(g) = f(g^{-1}), \quad g \in G.$$

Note that $\int_G \tilde{f}(g)dg = \int_X f(x) d\mu(x)$ where μ is the volume element on G/K.

We are now in a position to state and prove the main theorem of this section which is implicit in the work of Berenstein and Zalcman ([9]) and Berenstein and Shahshahani ([7]), though not stated in this form.

THEOREM 5.2. *A relatively compact measurable subset* E *of* X *of positive measure has the Pompeiu property if and only if*

$$\Pi_\lambda(1_{\tilde{E}}) \overset{\text{def}}{=} \int_{\tilde{E}} \Pi_\lambda(g)\, dg$$

is a nonzero operator for every $\lambda \in \mathbf{C}$.

Proof. Given $E \subseteq X$ as in the statement, we notice that for $f \in C^\infty(X)$, $\int_{gE} f = 0$ for all $g \in G$ is equivalent to the condition $\tilde{f} * \check{1}_{\tilde{E}} = 0$. Define

$$\mathscr{U} = \{h \in C^\infty(G) : h(gk) = h(g) \quad \text{for all} \quad g \in G, k \in K, \quad \text{and} \quad h * \check{1}_{\tilde{E}} = 0\}.$$

Notice that $\mathscr{U}$ is a closed subspace invariant under left translation by elements of G. Then E has the Pompeiu property if and only if $\mathscr{U} = \{0\}$. Writing $\mathscr{V} = \mathscr{U} \cap \mathscr{E}$ (i.e., $\mathscr{V}$ is the space of K-bi-invariant functions in $\mathscr{U}$), we claim that $\mathscr{U} = \{0\}$ if and only if $\mathscr{V} = \{0\}$. First, if $\mathscr{U} \neq \{0\}$, choose $f \in \mathscr{U}$ such that $f(e) \neq 0$ where e is the identity element of G. Now for such an f, define the function h by

$$h(x) = \int_K f(kx)dk\,, \quad x \in G\,.$$

Since $\mathscr{U}$ is translation-invariant, $h \in \mathscr{U}$. On the other hand h is K-bi-invariant. So $h \in \mathscr{V} \subseteq \mathscr{U}$. But $h(e) = f(e) \neq 0$. It is now easy to show that $\mathscr{V}$ is a variety and by Theorem 5.1, $\mathscr{V} \neq \{0\}$ if and only if for some $\lambda \in \mathbf{C}$, $\phi_\lambda \in \mathscr{V}$.

Suppose now $\Pi_\lambda(1_{\tilde{E}}) = 0$ for some $\lambda \in \mathbf{C}$. We have for all $g \in G$,

$$\Pi_\lambda(g)\Pi_\lambda(1_{\tilde{E}}) = \int_{\tilde{E}} \Pi_\lambda(gx)dx = 0\,.$$

Consequently,

$$\begin{aligned} <\Pi_\lambda(g)\Pi_\lambda(1_{\tilde{E}})1, 1> &= \int_{\tilde{E}} <\Pi_\lambda(gx)1, 1>\, dx \\ &= \phi_\lambda * \check{1}_{\tilde{E}}(g) = 0\,. \end{aligned}$$

So $\phi_\lambda \in \mathscr{V}$ and hence E does not have the Pompeiu property. Conversely, suppose E does not have the Pompeiu property. Then $\mathscr{V} \neq \{0\}$ and hence there exists $\lambda \in \mathbf{C}$ such that $\phi_\lambda \in \mathscr{V}$. Further, if $\lambda \in i\mathbf{R}$, then $\phi_\lambda(x) > 0$ for all $x \in G$ and so $\phi_\lambda \notin \mathscr{V}$ as E has positive Haar measure. Thus $\lambda \notin i\mathbf{R}$, and then it is known that the representation Π_λ is irreducible (see [7]) and we have, for all $g \in G$,

$$\begin{aligned} \phi_\lambda * \check{1}_{\tilde{E}}(g) &= <\Pi_\lambda(g)\Pi_\lambda(1_{\tilde{E}})1, 1> \\ &= <\Pi_\lambda(1_{\tilde{E}})1, \Pi_\lambda^*(g)1> \\ &= <\Pi_\lambda(1_{\tilde{E}})1, \Pi_{\bar{\lambda}}(g^{-1})1> \\ &= 0\,. \end{aligned}$$

Since $\bar{\lambda} \notin i\mathbf{R}$, $\Pi_{\bar{\lambda}}$ is an irreducible representation; and hence 1 is a cyclic vector for $\Pi_{\bar{\lambda}}$. Our identity now implies $\Pi_\lambda(1_{\tilde{E}})1 = 0$. But since Π_λ is an irreducible class-1 representation and $1_{\tilde{E}}$ is a right K-invariant function, it follows that $\Pi_\lambda(1_{\tilde{E}})F = 0$ for all $F \in L^2(K/M)$. (Here we use the general fact that if Π_λ is a class-1 principal series representation and h is a right K-invariant function, then $\Pi_\lambda(h)$ is completely determined by its action on the constant function 1 on K/M. In fact, $\Pi_\lambda(f) = 0$ on the orthogonal complement of 1 in $L^2(K/M)$.) Thus $\Pi_\lambda(1_{\tilde{E}}) = 0$, proving the theorem.

As in Euclidean spaces, it is not possible to verify the condition of Theorem 5.2 in very many cases. Theorem 5.2 may also be taken as a starting point for considering the differential equations formulation of the

Pompeiu problem as in Berenstein and Shahshahani [7]. We quote a main result of theirs (their Proposition 3 and Corollary 1).

THEOREM 5.3 (Berenstein and Shahshahani). *Let* Ω *be an open relatively compact subset of* $X = G/K$ *such that* $X - \Omega$ *is connected and* $\partial\Omega$ *is Lipschitz. Assume that* Ω *does not have the Pompeiu property. Then* $\partial\Omega$ *is analytic.*

It follows from Theorem 5.3 that, for instance geodesic triangles in X have the Pompeiu property. On the other hand, Berenstein and Zalcman ([9]) as well as Berenstein and Shahshahani ([7]) point out that geodesic balls in X do not have the Pompeiu property. In fact, Sitaram ([29]) proves that if E is a relatively compact K-bi-invariant set and $m(E) > 0$, then E does not have the Pompeiu property. As in $\mathbf{R}^n$, it remains an open question whether the only sets with smooth boundary and connected exterior that fail to have the Pompeiu property are the geodesic balls. Analogues of Theorem 4.1 are also available for certain symmetric spaces (see [3] and [8]).

6. SYMMETRIC SPACES OF THE COMPACT TYPE

Let X be a symmetric space of the compact type, i.e., X is of the form G/K where G is a connected compact semisimple Lie group and K a suitable closed subgroup. Then (G, K) is a so-called Riemannian symmetric pair. G/K is equipped with a canonical G-invariant measure (see [15] for details).

As in the previous section, we shall connect the Pompeiu property with certain representations of the group G. We need only unitary irreducible representations Π of G on finite-dimensional Hilbert spaces $\mathcal{H}$. Such a representation Π is said to be a class-1 representation with respect to K if there exists a vector $0 \neq v \in \mathcal{H}$, such that for all $k \in K, \Pi(k)v = v$. For (G, K) a Riemannian symmetric pair it is well known that an irreducible representation Π is either of class-1 (in which case v is unique upto scalar multiples) or Π does not admit any nonzero vector v such that $\Pi(k)v = v$ for all $k \in K$ (see [16], p. 412). We record two simple observations for a continuous unitary irreducible representation Π of G: For $f \in L^1(G)$, we denote by $\Pi(f)$ the operator $\Pi(f) = \int_G \Pi(x) f(x)\,dx$. So if f is either right or left K-invariant, then $\Pi(f) = 0$ provided Π is not of class-1. If Π is class-1 and $0 \neq v_0$ is a K-fixed vector for Π and f is right K-invariant, then $\Pi(f) = 0$ on $(\mathbf{C}v_0)^{\perp}$, i.e., $\Pi(f)$ is completely determined by its action on v_0. On the other hand, if f is left K-invariant, $\Pi(f)$ maps $\mathscr{H}$ into $\mathbf{C}\, v_0$. If, further, f is also K-bi-invariant, then $\Pi(f) v_0 = c_{\Pi,f} \cdot v_0$ where the constant $c_{\Pi,f}$ can be explicitly computed in terms of the so-called elementary spherical functions. In fact, fix v_0 such that $\| v_0 \| = 1$ and let

$$\phi_\Pi(x) = \langle \Pi(x) v_0, v_0 \rangle \; .$$

Then ϕ_Π is continuous, in fact, a real analytic K-bi-invariant function, called the elementary spherical function associated with Π and now we have $c_{\Pi,f} = \int_G \phi_\Pi(x) f(x)\, dx$.

The following simple result, which we view as the analogue of Theorems 3.1 and 5.2 for Euclidean and non-compact spaces respectively, is implicit in the work of Berenstein and Zalcman ([9]), though not stated in this form. For sake of completeness we give a sketch of the proof.

THEOREM 6.1. *Let* $X = G/K$ *be a symmetric space of the compact type where* G *and* K *are as above. A measurable subset* $E \subseteq X$ *of positive measure has the Pompeiu property (for the usual G-action on X) if and only if for all irreducible representation* Π *of class 1 of the group* G, $\Pi\, (1_{\tilde{E}}) \neq 0$ *(where* $\tilde{E} = \{g \colon gK \in E\}$).

Sketch of Proof: Suppose that E fails to have the Pompeiu property. So there exists $f \in C(X)$, $f \not\equiv 0$ such that $\int_{gE} f(x)\, dx = 0$ for all $g \in G$. As in Section 5, this means $\tilde{f} * \check{1}_{\tilde{E}} = 0$. Now $\tilde{f}$ is a nonzero right K-invariant function and let Π be an irreducible representation of G such that $\Pi(\tilde{f}) \neq 0$. By an easy Peter-Weyl argument such a Π exists. By our previous discussion then Π is a class-1 representation. We have

$$\Pi(\tilde{f} * \check{1}_{\tilde{E}}) = \Pi(\tilde{f}) \cdot \Pi(\check{1}_{\tilde{E}}) = 0 \tag{6.1}$$

We claim that $\Pi(\check{1}_{\tilde{E}}) = 0$. If not, since $\check{1}_{\tilde{E}}$ is left-invariant, by our earlier discussion again $\Pi(\check{1}_{\tilde{E}})\, \mathscr{H} = \mathbf{C} \cdot v_0$ where v_0 is a nonzero K-fixed vector. But then by equation (6.1), $\Pi(\tilde{f}) v_0 = 0$ and hence $\Pi(\tilde{f}) = 0$ as $\tilde{f}$ is right K-invariant, which is a contradiction. Thus $\Pi(\check{1}_{\tilde{E}}) = 0$. If now $\tilde{\Pi}(g) = \Pi'(g^{-1})$, i.e., the representation contragredient to Π, then $\tilde{\Pi}$ is again an irreducible unitary representation of class-1. We have $\tilde{\Pi}(1_{\tilde{E}}) = 0$ as desired. The converse assertion is in fact easier and follows from the Peter-Weyl theorem. This result is related to an analogue of the Wiener-Tauberian theorem for compact symmetric spaces ([30]).

In the special case when E is invariant under the left action of K as well the condition of the Theorem 6.1 reduces to a condition on the *spherical Fourier transform* of $1_{\tilde{E}}$. In fact, it follows from our earlier discussion (since

$1_{\tilde{E}}$ is K-bi-invariant) that for Π irreducible of class-1, $\Pi(1_{\tilde{E}}) = 0$ if and only if $<\Pi(1_{\tilde{E}})v_0, v_0> = 0$ where v_0 is a nonzero K-fixed vector. In terms of the elementary spherical function introduced earlier this is equivalent to saying

$$\int_G 1_{\tilde{E}}(g)\phi_\Pi(g)dg = 0 .$$

Thus using earlier notation this is the same as saying $c_{\Pi, 1_{\tilde{E}}} = 0$. Finally, for a left K-invariant set $E \subseteq X$, the condition of Theorem 6.1 is equivalent to demanding $c_{\Pi, 1_{\tilde{E}}} \neq 0$ for all Π of class-1.

As an application we consider the case when E is the geodesic ball B_r of radius r about the identity coset $eK \in X$. We work out the case of $S^2 = SO(3, \mathbf{R})/SO(2, \mathbf{R})$. The irreducible representations of class-1 of $SO(3, \mathbf{R})$ come from the decomposition

$$L^2(S^2) = \bigoplus_{k=0}^{\infty} \mathscr{H}_k$$

where $\mathscr{H}_k$ is the space of spherical harmonics of (homogeneous) degree k. On each $\mathscr{H}_k$ we have the irreducible representation

$$\Pi_k(g)f(x) = f(g^{-1}x) \quad f \in \mathscr{H}_k, \quad g \in SO(3, \mathbf{R}), \quad x \in S^2 .$$

We identify $(1, 0, 0) \in S^2$ with the identity coset $SO(2, \mathbf{R})$. The unique $SO(2, \mathbf{R})$-fixed function in $\mathscr{H}_k$ is given by $v_k(x_1, x_2, x_3) = P_k(x_1)$ where P_k is the Legendre polynomial of degree k (see [16], p. 404). Integrating first on the parallels orthogonal to $(1, 0, 0)$ and then with respect to θ where $x_1 = \cos\theta$, we get

$$c_{\Pi_k, 1_{\tilde{B}_r}} = \int_0^r P_k(\cos\theta)\sin\theta\, d\theta = P_{k+1}(r)$$

by the properties of Legendre polynomials.

Summarising this discussion, we get the following particular case of the general result of Berenstein and Zalcman ([9], Theorem 4) — where the computation is shown for a general rank 1 symmetric space of compact type.

PROPOSITION 6.2 (Berenstein and Zalcman). *A geodesic ball of radius* r *on* S^2 *has the Pompeiu property if and only if* $P_{k+1}(r) \neq 0$ *for* $k = 0, 1, 2, \cdots$.

See also the discussion in the section on compact groups in [24].

7. Pompeiu property for two-sided translations on groups

In this section we consider the Pompeiu problem in the setting where X is one of the groups $M(2)$ or $SL(2, \mathbf{R})$ and the group action is that of two-sided translations. We start with a general definition.

Definition 7.1. Let G be a locally compact unimodular group. A measurable relatively compact subset $E \subseteq G$ of positive Haar measure is said to have the (two-sided) Pompeiu property if for $f \in C(G)$,

$$\int_{g_1 E g_2} f(x) dx = 0 \quad \forall g_1, g_2 \in G$$

implies $f \equiv 0$ where the integration is with respect to the Haar measure.

Except for the two groups mentioned the problem in general appears to be untractable at the moment, the primary reason being that there has not been much progress with the related problem of two-sided spectral analysis. If, however, we consider only functions $f \in L^1(G) \cap C(G)$ the problem considered above becomes easier and some investigations have been made in this restricted set-up (see [20], [21], [24]). Our treatment of $M(2)$ and $SL(2, \mathbf{R})$ relies on the work of Weit ([32]) and Ehrenpreis and Mautner ([13], [14]) on spectral analysis and synthesis on $M(2)$ and $SL(2, \mathbf{R})$ respectively. (See also [4], [31] in this connection.)

§ 7.1. We introduce a class of representations of the group $M(2)$. As in Section 3, we write $M(2) = \{(x, \sigma) : x \in \mathbf{R}^2, \sigma \in SO(2, \mathbf{R})\}$ where $(x, \sigma) \cdot (x', \sigma') = (x + \sigma x', \sigma\sigma')$ is the group multiplication and an element (x, σ) acts on $y \in \mathbf{R}^2$ by the rule $(x, \sigma) y = \sigma y + x$. Let $\lambda \in \mathbf{C}, \lambda \neq 0$ and we define representations Π_λ on the space $L^2(S^1)$ where $S^1 \subseteq \mathbf{R}^2$ is the unit circle: for $(x, A) \in M(2)$,

$$\Pi_\lambda(x, A) f(w) = e^{i\lambda x \cdot w} f(A^{-1} w) \quad f \in L^2(S), \quad w \in S^1,$$

where $\cdot$ is the inner product in $\mathbf{R}^2$. The Π_λ's are related to the representations of $M(2)$ on the eigenspaces of the Laplacian Δ on $\mathbf{R}^2$ and are irreducible (see [16], p. 12). If λ is real, then Π_λ is seen to be a unitary representation. The only other irreducible unitary representation (up to equivalence) of $M(2)$ are the characters:

$$\chi_n(x, A) = e^{in\theta}, \quad x \in \mathbf{R}^2, A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where $n = 0, \pm 1, \pm 2, \cdots$.

We now come to our result for the group $M(2)$.

THEOREM 7.1. *A measurable relatively compact subset* $E \subseteq M(2)$ *of positive Haar measure has the Pompeiu property if and only if*

(i) *the operators* $\int_E \Pi_\lambda(x)dx \neq 0$ *for each* $\lambda \in \mathbf{C}, \lambda \neq 0$;

(ii) $\int_E \chi_n(x)dx \neq 0$ *for each integer* n;

where the integrals are with respect to the Haar measure on $M(2)$.

Proof. If Π is a continuous irreducible representation of G on a Hilbert space and $\Pi(1_E) = 0$, then for a suitably chosen matrix element $f: x \to \langle \Pi(x)v_1, v_2 \rangle$, we have $\int_{g_1 E g_2} f(x)dx = 0$ for all g_1, g_2. The *only if* part now follows. To prove the *if* part we consider the two-sided ideal of $C(M(2))$:

$$\mathcal{U} = \{f \in C(M(2)): \int_{g_1 E g_2} f dx = 0 \quad \text{for all} \quad g_1, g_2 \in M(2)\}.$$

Assume $\mathcal{U} \neq \{0\}$. We shall prove that either for some $n, \int_E \chi_n(x)dx = 0$, or for some $\lambda \neq 0, \int_E \Pi_\lambda(x)dx = 0$. Since $\mathcal{U} \neq \{0\}$, by Weit's theorem ([32], Theorem 1), either $\chi_n \in \mathcal{U}$ for some n or there exists $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2, \lambda_1^2 + \lambda_2^2 \neq 0$ such that the functions $g_\lambda(x, A) = e^{i(\lambda_1 x_1 + \lambda_2 x_2)}$, where $(x, A) \in M(2)\ (x=(x_1, x_2))$, belongs to $\mathcal{U}$. In case $\chi_n \in \mathcal{U}$ we immediately get the desired result. So let for λ as above, $g_\lambda \in \mathcal{U}$. As noted by Weit if $\mu = (\mu_1, \mu_2) \in \mathbf{C}^2$ and $\mu_1^2 + \mu_2^2 = \lambda_1^2 + \lambda_2^2$, then $g_\mu \in \mathcal{U}$. We choose $\mu_1 = \sqrt{\lambda_1^2 + \lambda_2^2} \cos\theta, \mu_2 = \sqrt{\lambda_1^2 + \lambda_2^2} \sin\theta$ for some $\theta \in \mathbf{R}$, so that

$$g_\mu(x, A) = \exp(i\sqrt{\lambda_1^2 + \lambda_2^2}\, x \cdot w), \quad (x, A) \in M(2),$$

where $w = (\cos\theta, \sin\theta)$. Since $g_\mu \in \mathcal{U}$ for all $\theta \in \mathbf{R}$ and since $\mathcal{U}$ is closed, we have $h \in \mathcal{U}$ where

$$h(x, A) = \int_{S^1} \exp(i\sqrt{\lambda_1^2 + \lambda_2^2}\, x \cdot w)dw, \quad (x, A) \in M(2).$$

But $h(x, A) = \langle \Pi_z(x, A)1, 1 \rangle, (x, A) \in M(2)$ where $z = \sqrt{\lambda_1^2 + \lambda_2^2}$ and so we have

$$\int_{g_1 E g_2} \langle \Pi_z(g)1, 1 \rangle \, dg = 0$$

for all $g_1, g_2 \in G$. This means

$$\int_E <\Pi_z(g_1)\Pi_z(g)\Pi_z(g_2)1, 1> dg = 0\,, \quad g_1, g_2 \in G\,.$$

By taking adjoints and using the fact $\Pi_z(g)^* = \Pi_{-\bar{z}}(g^{-1})$,

$$\int_E <\Pi_z(g)\Pi_z(g_2)1, \Pi_{-\bar{z}}(g_1^{-1})1> dg = 0\,, \quad g_1, g_2 \in G\,.$$

Since Π_z and $\Pi_{-\bar{z}}$ are irreducible, 1 is a cyclic vector in $L^2(S^1)$ for both these representation and it will, therefore, follow

$$\int_E <\Pi_z(g)u, v> dg = 0$$

for all $u, v \in L^2(S^1)$. This shows that $\Pi_z(1_E) = 0$ and the theorem is proved.

§ 7.2. For simplicity, we consider the group

$$PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{+1, -1\}$$

rather than $SL(2, \mathbf{R})$ itself. For each $\lambda \in \mathbf{C}$, $SL(2, \mathbf{R})$ has the so-called principal series representation Π_λ (which are continuous representations realised on a Hilbert space). It is well known, that but for a countable set of λ's Π_λ is irreducible. Thus the set of representations $\{\Pi_\lambda : \lambda \in \mathbf{C}, \Pi_\lambda \text{ irreducible}\}$ along with another countable family of representations called the discrete series and all irreducible finite-dimensional representations account for all so-called "topologically completely irreducible" Banach representations (upto equivalence) of $PSL(2, \mathbf{R})$. We now state our theorem for $PSL(2, \mathbf{R})$:

THEOREM 7.2. *A relatively compact measurable subset* $E \subseteq PSL(2, \mathbf{R})$ *of positive Haar measure has the Pompeiu property if and only if for each topologically completely irreducible Banach representation* Π *of* $PSL(2, \mathbf{R})$, $\Pi(1_E) \neq 0$.

Sketch of Proof. As with Theorem 7.1, the proof relies on spectral analysis of two-sided ideals of $C^\infty(G)$ due to Ehrenpreis and Mautner ([13], [14]): If $0 \neq \mathcal{U} \subseteq C^\infty(G)$ is a closed two-sided ideal in $C^\infty(G)$, then there exists $\lambda \in \mathbf{C}$ and a nontrivial function of the form $\phi^\lambda_{n,m} : x \to <\Pi_\lambda(x)e_n, e_m>$ is in $\mathcal{U}$. Here $\{e_j\}$ is an orthonormal basis of $\mathcal{H}_\lambda$, the representation space for Π_λ and moreover, $\Pi_\lambda(k)e_j = \chi_j(k)e_j$ where $\{\chi_j\}$ are the characters of the maximal compact subgroup $SO(2, \mathbf{R})/\{+1, -1\}$ of $PSL(2, \mathbf{R})$ and $k \in SO(2, \mathbf{R})/\{+1, -1\}$. To prove the theorem, we define $\mathcal{U}$ as in Theorem 7.1. If $\mathcal{U} \neq \{0\}$, then appealing to the result quoted above we get $\phi^\lambda_{n,m} \in \mathcal{U}$ for some $\lambda \in \mathbf{C}$ and n, m integers. If the corresponding Π_λ is irreducible, then the argument is as in Theorem 7.1 and we can prove $\Pi_\lambda(1_E) = 0$. If Π_λ is not irreducible, then depending on n, m we can find a discrete series representation or an irreducible finite dimensional representation Π occurring either as a subrepresentation or as a subquotient of Π_λ for which $\Pi(1_E) = 0$. To do this, we need the exact G-module structure of Π_λ which in the case of $PSL(2, \mathbf{R})$ is available (see for example [14]).

§ 7.3. Consider the case $G = \mathbf{R}^2$ and G acting on itself *by translations*. In this case, Brown, Schreiber and Taylor have proved that there are no Pompeiu sets ([12]). In view of this it would be natural to ask if there are sets E satisfying the conditions of Theorem 7.1 at all. Identify $\mathbf{R}^2$ with G/K where $G = M(2)$ and $K = SO(2, \mathbf{R})$; if $E \subseteq G/K$ then one can show that the condition of Brown, Schreiber and Taylor considered in Section 3 is equivalent to the condition $\Pi_\lambda(1_{\tilde{E}}) \neq 0$ for $\lambda \in \mathbf{C}, \lambda \neq 0$. (A special case of this observation is also made in [30]). Hence by the discussion in Section 3, there are plenty of sets E with this property. As we have seen, topologically $G \approx \mathbf{R}^2 \times SO(2, \mathbf{R})$. We now observe that if E is chosen as above in $\mathbf{R}^2$ and I is a suitably chosen arc in $SO(2, \mathbf{R})$, then the set $F = E \times I$ considered as a subset of G satisfies $\Pi_\lambda(1_F) \neq 0$ for all $\lambda \in \mathbf{C} - \{0\}$ as well as $\chi_n(1_F) \neq 0$ for all $n \in \mathbf{Z}$. (We need only to choose I whose length is irrational modulo 2π.)

8. Concluding remarks

In this paper, we have restricted our attention to the Pompeiu property for a single set E. One can also consider the Pompeiu property for a collection of sets or distributions of compact support as in [9], [12]. There are also closely related properties such as the Morera property — see [12] for details.

As pointed out earlier the Pompeiu problem becomes easier if one considers only integrable functions. Investigations under this assumption have been done, for example, in [2], [20], [24] and [28]. If one only considers integrable functions one need not restrict oneself to relatively compact sets. Moreover, considering integrable functions is equivalent to considering finite complex measures. Thus for G a locally compact abelian group a Borel subset $E \subseteq G$ is said to be a *determining set for finite complex measures* if for a finite complex measure μ on G, $\mu(gE) = 0$ for all $g \in G$ implies $\mu = 0$.

For locally compact abelian groups it is easy to see that a set of finite Haar measure is a determining set for finite complex measures if and only if the Fourier transform $\hat{1}_E$ does not vanish on any nonempty open subset of the dual group $\hat{G}$. Thus bounded Borel subsets of $\mathbf{R}^n$ of positive Lebesgue measure are determining sets by the analyticity of $\hat{1}_E$. Classical quasi-analyticity results apply to give conditions on the growth of an unbounded subset $E \subseteq \mathbf{R}^n$ to be a determining set. Settling a problem that was open for some time, Kargaev ([17]) proved the existence of sets $E \subseteq \mathbf{R}^n$ of finite Lebesgue measure which are not determining sets for finite complex measures.

The problem of determining sets has also been studied with the class of probability measures replaced by other classes of measures, e.g., a class of infinite measures with growth/decay conditions (see [22], [11] and [28]). Also different groups of homeomorphisms acting on X have been considered in these studies.

Finally, we refer to the following form of the *support problem* analogous to the well known problem in the case of Radon transform. Let X be a symmetric space (Euclidean, compact or non-compact). Let x_0 be a fixed point of X. If E is a relatively compact subset of positive measure and if $\int_{gE} f = 0$ for all $g \in G$ with $d(x_0, gx_0) > R$ what can one say about the support of f with respect to the reference point x_0? (Here, d stands for the geodesic distance.) Some partial answers to this question are known (see [26] and [28]).

We have not addressed ourselves in this paper to the situation when X is an infinite-dimensional Hilbert space or X is an arbitrary Riemannian manifold. Another important problem we have not considered is the *local version* of the Pompeiu problem. (For this, we refer the reader to [5] and [6]). We have restricted ourselves to the situation of symmetric spaces and locally compact groups and the relationship of the Pompeiu problem with harmonic analysis and representation theory.

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