

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 36 (1990)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE POMPEIU PROBLEM REVISITED  
**Autor:** Bagchi, S. C. / Sitaram, A.  
**Kapitel:** 2. Spectral analysis of radial functions  
**DOI:** <https://doi.org/10.5169/seals-57903>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 09.03.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

provide a brief survey of the literature on some allied problems. Though extensive, our bibliography is far from complete. We refer the reader to the bibliographies in [9], [12] and [35].

## 2. SPECTRAL ANALYSIS OF RADIAL FUNCTIONS

We denote by  $\mathcal{E}(\mathbf{R}^n)$ , the space of  $C^\infty$  functions on  $\mathbf{R}^n$  with the usual topology and by  $\mathcal{E}'(\mathbf{R}^n)$ , the dual space of distributions of compact support with the strong topology — both Fréchet-Montel and hence reflexive spaces.  $C_c^\infty(\mathbf{R}^n)$  is the space of  $C^\infty$ -functions of compact support. For a space of functions or distributions  $\mathcal{F}$ , we denote the usual action of an element  $\sigma$  of the orthogonal group  $O(n, \mathbf{R})$  by the notation  $f \rightarrow f^\sigma$ .  $\mathcal{F}_{\text{rad}}$  will stand for the space of those  $f \in \mathcal{F}$  which are invariant under  $O(n, \mathbf{R})$ , i.e.,  $f^\sigma = f$  for all  $\sigma \in O(n, \mathbf{R})$ .  $\mathcal{E}'(\mathbf{R}^n)_{\text{rad}}$  is a closed subspace of  $\mathcal{E}'(\mathbf{R}^n)$  and the spaces  $\mathcal{E}(\mathbf{R}^n)_{\text{rad}}$  and  $\mathcal{E}'(\mathbf{R}^n)_{\text{rad}}$  are (strong) duals of each other. In the case  $n = 1$ , even functions are the analogues of radial functions and we write  $\mathcal{F}_e$  to mean  $\mathcal{F}_{\text{rad}}$ . Though our considerations in this section hold for all  $n \geq 2$ , we shall restrict ourselves to the case  $n = 2$  to keep the exposition simple.

We start with a slightly weaker version of the classical theorem of L. Schwartz ([23]).

**THEOREM 2.1** (L. Schwartz's theorem on spectral analysis). *Let  $\mathcal{U}$  be a nontrivial closed subspace of  $\mathcal{E}(\mathbf{R})$ , which is closed under translations, then  $\mathcal{U}$  contains an exponential function  $e^{i\lambda x}$  for some  $\lambda \in \mathbf{C}$ .*

As pointed out in [1] an immediate corollary of the theorem is: If  $\mathcal{U}$  is a nontrivial closed subspace of  $\mathcal{E}(\mathbf{R})_e$  which is closed under convolution against all  $T \in \mathcal{E}'(\mathbf{R})_e$ , then  $\mathcal{U}$  contains a function of the form  $\psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$ .

We now introduce a family of functions on  $\mathbf{R}^2$  which is central to spectral analysis of radial functions. For  $\lambda \in \mathbf{C}$ , define

$$\phi_\lambda(x) = \int_{|w|=1} e^{-i\lambda(x \cdot w)} dw, \quad x \in \mathbf{R}^2$$

where the integral is with respect to the normalised Lebesgue measure on the unit circle. Here  $x \cdot w$  is the usual inner product. It is immediate that  $\phi_\lambda$  is a radial function for each  $\lambda \in \mathbf{C}$ . For  $f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}}$ , we define a transform (sometimes called the Bessel transform):

$$\mathcal{G}f(\lambda) = \int_{\mathbf{R}^2} \phi_\lambda(x) f(x) dx, \quad \lambda \in \mathbf{C}.$$

We see that if  $\hat{f}$  is the Fourier-Laplace transform of  $f \in C_c^\infty(\mathbf{R}^2)$ , i.e.,

$$\hat{f}(z_1, z_2) = \int_{\mathbf{R}^2} e^{-i(z \cdot x)} f(x) dx, \quad z = (z_1, z_2) \in \mathbf{C}^2,$$

then we have

$$(2.1) \quad \mathcal{G}f(\lambda) = \hat{f}(\lambda, 0), \quad \lambda \in \mathbf{C}, \quad f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}}.$$

Both the transforms  $\mathcal{G}$  defined above and the Fourier-Laplace transform have their obvious extension to  $\mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$ . We have for  $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$ ,

$$\begin{aligned} \mathcal{G}T(\lambda) &= T(\phi_\lambda), \quad \lambda \in \mathbf{C} \\ \hat{T}(z_1, z_2) &= T(e_z), \quad z \in \mathbf{C}^2 \end{aligned}$$

where  $e_z(x) = e^{-i(z_1 x_1 + z_2 x_2)} = e^{-i(z \cdot x)}$ . We again have

$$\mathcal{G}T(\lambda) = \hat{T}(\lambda, 0), \quad \lambda \in \mathbf{C}.$$

By applying the Paley-Wiener theorem we are able to obtain a description of the function space  $\mathcal{X} = \{\mathcal{G}T : T \in C_c^\infty(\mathbf{R}^2)_{\text{rad}}\}$ .

LEMMA 2.2.  $\mathcal{X}$  is the space of even entire functions  $f$  on  $\mathbf{C}$  such that for some constants,  $c, N$  and  $A$  (depending on  $f$ ),

$$|f(\lambda)| \leq C(1 + |\lambda|)^N e^{A|\text{Im } \lambda|}, \quad \lambda \in \mathbf{C}.$$

*Proof.* By the Paley-Wiener theorem an entire function  $\phi = \hat{T}$  for some  $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$  if and only if for some  $C, N, A > 0$ ,

$$|\phi(z)| \leq C(1 + |z|)^N e^{A|\text{Im } z|}$$

and

$$\phi(z) = \phi(\sigma z)$$

for all  $z = (z_1, z_2) \in \mathbf{C}^2$  and  $\sigma \in SO(2, \mathbf{R})$  (here  $\text{Im } z = (\text{Im } z_1, \text{Im } z_2)$ ). The latter condition is equivalent to saying  $\phi(z) = \phi(z')$  whenever  $z_1^2 + z_2^2 = z_1'^2 + z_2'^2$ . To see this, consider, for each  $\alpha \in \mathbf{C}$ ,

$$M_\alpha = \{z : z_1^2 + z_2^2 = \alpha^2\}.$$

If  $\alpha \neq 0$ ,  $M_\alpha$  is a connected analytic submanifold of  $\mathbf{C}^2$  of complex dimension 1 and  $SO(2, \mathbf{R})(\alpha, 0, \dots, 0)$  is a real submanifold of  $M_\alpha$  of dimension 1 on which the analytic function  $\phi$  is given to be constant. This forces  $\phi$  to be a constant on  $M_\alpha$ . A modification of the argument is necessary for  $\alpha = 0$ .

The lemma now follows from the simple observation that if  $\lambda^2 = z_1^2 + z_2^2$ , then  $(\text{Im } \lambda)^2 \leq (\text{Im } z_1)^2 + (\text{Im } z_2)^2$  and from the relation 2.1.

A straight-forward application of the one-dimensional Paley-Wiener theorem for even distributions of compact support will show that  $\mathcal{X}$  is also equal to

$$\{\hat{T}: T \in \mathcal{E}'(\mathbf{R})_e\}.$$

This identification allows us to define the linear map  $\Sigma$  by

$$\begin{aligned} \Sigma: \mathcal{E}'(\mathbf{R}^2)_{\text{rad}} &\rightarrow \mathcal{E}'(\mathbf{R}^2)_e \\ (\Sigma T)^\wedge(\lambda) &= \mathcal{G}T(\lambda), \quad \lambda \in \mathbf{C}. \end{aligned}$$

$\Sigma$  is one-to-one and onto. Moreover, we have the following description of the strong topology in  $\mathcal{E}'(\mathbf{R}^n)$  (see [12], prop. 2.1):  $T_n \rightarrow T$  if and only if

- (i)  $\hat{T}_n \rightarrow \hat{T}$  uniformly on compact sets along with the derivatives and
- (ii)  $\hat{T}_n, n \geq 1$  satisfy the uniform Paley-Wiener condition:

$$|\hat{T}_n(z)| \leq C(1+|z|)^N e^{A|\text{Im } z|}, \quad z \in \mathbf{C}^n.$$

for some  $C, N, A > 0$ . This description coupled with the observation in the last step of the proof of Lemma 2.2 gives that  $\Sigma$  is a topological linear isomorphism between  $\mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$  and  $\mathcal{E}'(\mathbf{R})_e$  preserving convolution.

On using the reflexivity of  $\mathcal{E}(\mathbf{R})_e$  and  $\mathcal{E}(\mathbf{R}^2)_{\text{rad}}$  we now get the map  $\tilde{\Sigma}$ :

$$\begin{aligned} \tilde{\Sigma}: \mathcal{E}(\mathbf{R}^2)_{\text{rad}} &\rightarrow \mathcal{E}(\mathbf{R})_e, \\ \langle \Sigma(T), \tilde{\Sigma}(f) \rangle &= \langle T, f \rangle \quad T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}, f \in \mathcal{E}(\mathbf{R}^2)_{\text{rad}} \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  is the pairing of dual spaces.

We now have the following useful lemma:

LEMMA 2.3. *With the notation above, we have*

$$\Sigma(T) * \tilde{\Sigma}(f) = \tilde{\Sigma}(T * f)$$

for all  $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$  and  $f \in \mathcal{E}(\mathbf{R}^2)_{\text{rad}}$ , where  $*$  denotes the usual convolution on  $\mathbf{R}^2$ .

*Proof.* Let  $S \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$ . Consider

$$\langle \Sigma(S), (\Sigma(T) * \tilde{\Sigma}(f)) \rangle$$

$$\begin{aligned}
&= \Sigma(S) * (\Sigma(T) * \tilde{\Sigma}(f))^\vee(0) \\
&\quad (\text{where } g^\vee(x) = g(-x), g \in \mathcal{E}(\mathbf{R}^2), x \in \mathbf{R}^2), \\
&= \Sigma(S) * ((\tilde{\Sigma}f)^\vee * \Sigma(T)^\vee)(0) \\
&= \Sigma(S) * (\tilde{\Sigma}(f) * \Sigma(T)) \text{ as } \Sigma(T), \tilde{\Sigma}(f) \text{ are even} \\
&= \Sigma(S) * (\Sigma(T) * \tilde{\Sigma}(f))(0) \\
&= \langle \Sigma(S * T), \tilde{\Sigma}f \rangle \\
&\quad (\text{using } \Sigma S * \Sigma T = \Sigma S * T) \\
&= \langle S * T, f \rangle \\
&= S * T * f(0) \text{ as } f \text{ is even} \\
&= \langle S, T * f \rangle .
\end{aligned}$$

On the other hand,

$$\langle \Sigma(S), \tilde{\Sigma}(T * f) \rangle = \langle S, T * f \rangle .$$

The lemma is proved.

Finally, we come to the main result of the section: the spectral analysis theorem for radial functions. As we remarked in the introduction, the development in this section is along the same lines as in [1] where the corresponding result for rank-1 non-compact symmetric spaces is proved.

**THEOREM 2.4.** *Let  $\mathcal{V}$  be a closed nonzero subspace of  $\mathcal{E}(\mathbf{R}^2)_{\text{rad}}$  such that for all  $T \in \mathcal{E}'(\mathbf{R}^2)_{\text{rad}}$  and  $f \in \mathcal{V}$ ,  $T * f \in \mathcal{V}$ . Then there exists  $\lambda \in \mathbf{C}$  such that  $\phi_\lambda \in \mathcal{V}$ .*

*Proof.* Consider the closed and nontrivial subspace  $M$  of  $\mathcal{E}(\mathbf{R})_e$  such that  $\tilde{\Sigma}(\mathcal{V}) = M$ . By Lemma 2.3,  $M$  is closed under convolution with elements  $S \in \mathcal{E}'(\mathbf{R})_e$ . By the remarks following Theorem 2.1 now, there exists  $\lambda \in \mathbf{C}$  such that the functions  $\Psi_\lambda \in M$ , where  $\Psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$ ,  $x \in \mathbf{R}$ . A simple calculation now shows

$$\langle \phi_\lambda, f \rangle = \langle \Psi_\lambda, \Sigma f \rangle \quad f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}} \subseteq \mathcal{E}'(\mathbf{R}^2)_{\text{rad}} .$$

Thus  $\tilde{\Sigma}\phi_\lambda = \Psi_\lambda$  and hence  $\phi_\lambda \in \mathcal{V}$ .

### 3. POMPEIU PROBLEM FOR THE $M(2)$ ACTION ON $\mathbf{R}^2$

The Euclidean motion group  $M(2)$  is the semidirect product of  $\mathbf{R}^2$  with the rotation group  $SO(2, \mathbf{R})$ .