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$$= \Sigma(S) * \left(\Sigma(T)*\tilde{\Sigma}(f)\right)^{\vee}(0)$$
$$\left(\text{where } g^{\vee}(x) = g(-x), g \in \mathscr{E}(\mathbf{R}^2), x \in \mathbf{R}^2\right),$$
$$= \Sigma(S) * \left((\tilde{\Sigma} f)^{\vee} * \Sigma(T)^{\vee}\right)(0)$$
$$= \Sigma(S) * \left(\tilde{\Sigma}(f)*\Sigma(T)\right) \text{ as } \Sigma(T), \tilde{\Sigma}(f) \text{ are even}$$
$$= \Sigma(S) * \left(\Sigma(T)*\tilde{\Sigma} f\right)(0)$$
$$= <\Sigma(S*T), \tilde{\Sigma} f>$$
$$(\text{using } \Sigma S * \Sigma T = \Sigma S * T)$$
$$= <S*T, f>$$
$$= S * T * f(0) \text{ as} f \text{ is even}$$
$$= <S, T*f> .$$

On the other hand,

$$<\Sigma(S), \tilde{\Sigma}(T*f)> = <S, T*f> .$$

The lemma is proved.

Finally, we come to the main result of the section: the spectral analysis theorem for radial functions. As we remarked in the introduction, the development in this section is along the same lines as in [1] where the corresponding result for rank-1 non-compact symmetric spaces is proved.

THEOREM 2.4. *Let* $\mathscr{V}$ *be a closed nonzero subspace of* $\mathscr{E}(\mathbf{R}^2)_{\text{rad}}$ *such that for all* $T \in \mathscr{E}'(\mathbf{R}^2)_{\text{rad}}$ *and* $f \in \mathscr{V}, T * f \in \mathscr{V}$. *Then there exists* $\lambda \in \mathbf{C}$ *such that* $\phi_\lambda \in \mathscr{V}$.

Proof. Consider the closed and nontrivial subspace M of $\mathscr{E}(\mathbf{R})_e$ such that $\tilde{\Sigma}(\mathscr{V}) = M$. By Lemma 2.3, M is closed under convolution with elements $S \in \mathscr{E}'(\mathbf{R})_e$. By the remarks following Theorem 2.1 now, there exists $\lambda \in \mathbf{C}$ such that the functions $\Psi_\lambda \in M$, where $\Psi_\lambda(x) = (e^{i\lambda x} + e^{-i\lambda x})/2$, $x \in \mathbf{R}$. A simple calculation now shows

$$<\phi_\lambda, f> = <\Psi_\lambda, \Sigma f> \quad f \in C_c^\infty(\mathbf{R}^2)_{\text{rad}} \subseteq \mathscr{E}'(\mathbf{R}^2)_{\text{rad}} .$$

Thus $\tilde{\Sigma}\phi_\lambda = \Psi_\lambda$ and hence $\phi_\lambda \in \mathscr{V}$.

3. POMPEIU PROBLEM FOR THE $M(2)$ ACTION ON $\mathbf{R}^2$

The Euclidean motion group $M(2)$ is the semidirect product of $\mathbf{R}^2$ with the rotation group $SO(2, \mathbf{R})$.

$$M(2) = \{(x, \sigma): x \in \mathbf{R}^2, \sigma \in SO(2, \mathbf{R})\}$$

where

$$(x, \sigma) \cdot (x', \sigma') = (x + \sigma x', \sigma\sigma')$$

is the group multiplication and an element (x, σ) acts on $y \in \mathbf{R}^2$ by the rule $(x, \sigma)y = \sigma y + x$.

Let E be a relatively compact subset of $\mathbf{R}^2$ of positive Lebesgue measure. If $f \in C(\mathbf{R}^2)$, the space of continuous functions on $\mathbf{R}^2$, the vanishing of the integrals

$$\int_{gE} f(x)dx = 0\,, \quad \textit{for all} \quad g \in M(2)$$

$$\text{i.e.} \quad \int_{\sigma E + y} f(x)dx = 0\,, \quad \textit{for all} \quad \sigma \in SO(2, \mathbf{R})\,, \quad y \in \mathbf{R}^2$$

can be restated as $f * \check{1}_{\sigma E} \equiv 0$, *for all* $\sigma \in SO(2, \mathbf{R})$ or, equivalently $f^\sigma * \check{1}_E \equiv 0$ for all $\sigma \in SO(2, \mathbf{R})$, where $f^\sigma(x) = f(\sigma x)$ and $\check{1}_E(x) = 1_E(-x)$, $x \in \mathbf{R}^2$. We write

$$\mathscr{U} = \{f \in \mathscr{E}(\mathbf{R}^2): f^\sigma * \check{1}_E = 0 \quad \textit{for all} \quad \sigma \in SO(2, \mathbf{R})\}\,.$$

From elementary smoothing arguments, it follows that E has the Pompeiu property if and only if $\mathscr{U} = \{0\}$. $\mathscr{U}$ is a closed subspace of $\mathscr{E}(\mathbf{R}^2)$ which is invariant under translation and rotation. Let again

$$\mathscr{V} = \{f \in \mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}}: f * \check{1}_E = 0\}\,.$$

Then $\mathscr{V} \subseteq \mathscr{U}$, $\mathscr{V}$ is a closed subspace of $\mathscr{E}(\mathbf{R}^2)_{\mathrm{rad}}$ and $T * \mathscr{V} \subseteq \mathscr{V}$ for all $T \in \mathscr{E}'(\mathbf{R}^2)_{\mathrm{rad}}$.

We now prove the main theorem of [12] mentioned in the Introduction. (However, we restrict ourselves to indicator functions of sets, rather than general distributions of compact support.)

THEOREM 3.1 (Brown, Schreiber and Taylor). *A relatively compact subset* $E \subseteq \mathbf{R}^2$ *of positive Lebesgue measure does* not *have the Pompeiu property if and only if there exists* $\alpha \in \mathbf{C}$, $\alpha \neq 0$ *such that*

$$\hat{1}_E(z_1, z_2) = 0 \quad \textit{whenever} \quad z_1^2 + z_2^2 = \alpha^2\,,$$

where $\hat{1}_E$ *is the Laplace-Fourier transform of the characteristic function* 1_E *of* E.

Proof. The *if* part is immediate; for instance, take any $z = (z_1, z_2)$ such that $z_1^2 + z_2^2 = \alpha^2$ and consider the function $e^{iz \cdot x}$. To prove the *only if* part, suppose E has the Pompeiu property. Let $\mathscr{U}$ and $\mathscr{V}$ be defined as above; by assumption we have $\mathscr{U} \neq \{0\}$. We shall now prove that $\mathscr{V} \neq \{0\}$. Choose $f \in \mathscr{U}$ with $f(0) \neq 0$ (this is possible as $\mathscr{U}$ is translation-invariant). Define

$$h(y) = \int_{SO(2,\mathbf{R})} f(\sigma y) d\sigma , \quad y \in \mathbf{R}^2 .$$

As $\mathscr{U}$ is $SO(2, \mathbf{R})$-invariant, the function $h \in \mathscr{U}$. Further, h is a radial function by definition, so $h \in \mathscr{V}$. But then $h(0) = f(0) \neq 0$. Thus $\mathscr{V} \neq \{0\}$ and by Theorem 2.2, we have $\lambda \in \mathbf{C}$ such that $\phi_\lambda \in \mathscr{V}$. Further, ϕ_0 is the constant function and hence ϕ_0 cannot belong to $\mathscr{V} \subseteq \mathscr{U}$. So $\lambda \neq 0$ and $\phi_\lambda \in \mathscr{V}$ and, in particular, $\phi_\lambda * \check{1}_E(0) = 0$. In the notation of Section 2, this means $\mathscr{G}1_E(\lambda) = 0$ and hence $\hat{1}_E(\lambda, 0) = 0$. The $SO(2, \mathbf{R})$-invariance of $\mathscr{U}$ now shows that $\hat{1}_E$ vanishes on $SO(2, \mathbf{R}) . (\lambda, 0)$. The analyticity argument in Lemma 2.2 will now prove that $\hat{1}_E$ vanishes at all (z_1, z_2) where $z_1^2 + z_2^2 = \lambda^2$. This proves the theorem.

The condition in Theorem 3.1 can also be given a representation theoretic interpretation in terms of the so-called class-1 principal series representation of $M(2)$ — see Section 7.3 for a more precise statement. As we remarked earlier, the condition of the theorem is verifiable only for sets having strong geometric properties. We quote two results from [12] without proof.

THEOREM 3.2 (Brown, Schreiber and Taylor). *The ellipse*

$$E = \left\{ (x, y) \in \mathbf{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1 \right\}$$

has the Pompeiu property if and only if $a, b > 0$ *and* $a \neq b$.

When $a = b > 0$, D is the disc and we have

$$\hat{1}_D(z_1, z_2) = \text{const. } J_1(\sqrt{z_1^2 + z_2^2}) \ / \ \sqrt{z_1^2 + z_2^2} , \quad (z_1, z_2) \in \mathbf{C}^2 ,$$

where J_1 is the Bessel function. Since there are infinitely many zeros of J_1, D does not have the Pompeiu property. The next theorem, obtained through a careful estimate ([12]) needs a definition.

Definition 3.3. Let $\Gamma = \Gamma(t)$, $-1 \leqslant t \leqslant 1$ be a Lipschitz curve in $\mathbf{R}^2$ with well defined (a.e.) unit tangent vectors $T(t) = \Gamma'(t) / |\Gamma'(t)|$. The point $p = \Gamma(0)$ is a corner of Γ if both the right and the left limits of $T(t)$ as $t \to 0$ exist and are not multiples of each other.

THEOREM 3.4 (Brown, Schreiber and Taylor). *Let Ω be a compact connected subset of $\mathbf{R}^2$. Suppose that there is a half-plane H and a unique point $p \in \Omega \cap H$ of maximal distance from the boundary ∂H of H. If the boundary of Ω near p is given by a Lipschitz curve with a corner at p then Ω has the Pompeiu property.*

Let now Ω be a bounded Borel subset of the plane of positive measure and suppose that Ω does not have the Pompeiu property. By Theorem 3.1, $\hat{1}_\Omega$ vanishes on the algebraic variety $M_\alpha = \{(z_1, z_2): z_1^2 + z_2^2 = \alpha^2\}$ for some $\alpha \neq 0$. As observed in [33] and [34], $\hat{1}_\Omega/(z_1^2+z_2^2-\alpha^2)$ is now an entire function and standard Paley-Wiener theorem yields the following proposition.

PROPOSITION 3.5. *If Ω is a bounded Borel subset of $\mathbf{R}^2$ of positive measure with $\hat{1}_\Omega$ vanishing on $M_\alpha, \alpha \neq 0,$ then the function $g(z_1, z_2) = \hat{1}_\Omega/(z_1^2 + z_2^2 - \alpha^2)$ is an entire function on $\mathbf{C}^2$ which is the Laplace-Fourier transform of a distribution of compact support.*

Proposition 3.5 immediately gives rise to a partial differential equation. For, if T is the distribution whose Fourier transform is g, then from

$$(z_1^2+z_2^2-\alpha^2)g(z_1, z_2) = \hat{1}_\Omega(z_1, z_2)$$

we have

$$\Delta T + \alpha^2 T = -1_\Omega \tag{3.1}$$

where Δ is the Laplacian. Conversely, if there exists a distribution T of compact support satisfying the equation (3.1), then $\hat{1}_\Omega$ vanishes on M_α and hence Ω does not have the Pompeiu property. We also remark that if Ω is, further, a bounded simply connected open set and the equation (3.1) has a solution, then α^2 is necessarily a positive real number as can be seen from a simple Green's theorem argument (see [34] for a proof). The equation has been studied in [3], [33] and [34]. In [33] it was proved that a solution of (3.1), if it exists is actually a function. We shall discuss some more of these results in the next section. We end the present section by quoting the main theorem of [34]. This result extends Theorem 3.4 and, barring sets of rotational symmetry all known sets failing to have the Pompeiu property are covered by this result. For a bounded subset $\Omega \subseteq \mathbf{R}^2$, we denote by $\partial^*\Omega$ the boundary of the unbounded component.

THEOREM 3.5 (S. A. Williams). *Let* Ω *be a bounded open subset such that the equation* $\Delta T + \alpha^2 T = -1_\Omega$ *has a function solution of compact support for some* $\alpha > 0$. *Let* R, K, L *be positive real numbers such that* $L > KR$. *Assume that for* $P \in \partial^*\Omega$ *there exists a coordinate system* (x, y) *around* P *so that*

(i) $Q = (-R, R) \times (-L, L)$ *intersects* $\partial\Omega$ *in the graph* $y = f(x)$ *of a Lipschitz function* f *with Lipschitz constant* K, *and*

(ii) $Q \cap \Omega = \{(x, y) : |x| < R$ *and* $f(x) < y < L\}$.

Then f *is real analytic in a neighbourhood of* P.

Thus if we restrict ourselves to the class $\mathscr{D}$ of simply connected bounded open sets with Lipschitz boundary then $\Omega \in \mathscr{D}$ can fail to have the Pompeiu property only if $\partial\Omega$ is real analytic.

4. A LONG-STANDING CONJECTURE !

The following Conjecture has received quite some attention in the literature ([3], [10], [34]).

Conjecture. If $\Omega \subseteq \mathbf{R}^2$ is in the class $\mathscr{D}$ described above and if Ω does not have the Pompeiu property, then Ω is a disc.

As pointed out before, the work of Williams shows that is is enough to consider Ω with $\partial\Omega$ real analytic. For $\Omega \in \mathscr{D}$, the existence of (a necessarily positive) α^2 for which (3.1) has a distribution solution of compact support is equivalent to the existence of a positive γ for which the following overdetermined system has a solution.

$$\text{(4.1)} \qquad \Delta T + \gamma T = 0 \quad \text{on} \quad \Omega$$

$$T = \text{constant} \neq 0 \quad \text{on} \quad \partial\Omega, \ \partial T/\partial n \equiv 0 \quad \text{on} \quad \partial\Omega$$

(see [34] for details). Thus the conjecture can be stated as follows:

If for $\Omega \in \mathscr{D}$, there exists $\gamma > 0$ for which (4.1) admits a solution, then Ω is a disc.

It is remarked in [34] that the conjecture is closely related to a result of Serrin ([25]): If Ω is a bounded connected open set with smooth boundary on which

$$\Delta u = -1 \quad \text{on} \quad \Omega$$

$$u = 0, \ \partial u/\partial n = \text{constant} \quad \text{on} \quad \partial\Omega$$

has a function solution, then Ω must be a disc.