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Pompeiu problem as in Berenstein and Shahshahani [7]. We quote a main result of theirs (their Proposition 3 and Corollary 1).

THEOREM 5.3 (Berenstein and Shahshahani). *Let* Ω *be an open relatively compact subset of* $X = G/K$ *such that* $X - \Omega$ *is connected and* $\partial\Omega$ *is Lipschitz. Assume that* Ω *does not have the Pompeiu property. Then* $\partial\Omega$ *is analytic.*

It follows from Theorem 5.3 that, for instance geodesic triangles in X have the Pompeiu property. On the other hand, Berenstein and Zalcman ([9]) as well as Berenstein and Shahshahani ([7]) point out that geodesic balls in X do not have the Pompeiu property. In fact, Sitaram ([29]) proves that if E is a relatively compact K-bi-invariant set and $m(E) > 0$, then E does not have the Pompeiu property. As in $\mathbf{R}^n$, it remains an open question whether the only sets with smooth boundary and connected exterior that fail to have the Pompeiu property are the geodesic balls. Analogues of Theorem 4.1 are also available for certain symmetric spaces (see [3] and [8]).

6. SYMMETRIC SPACES OF THE COMPACT TYPE

Let X be a symmetric space of the compact type, i.e., X is of the form G/K where G is a connected compact semisimple Lie group and K a suitable closed subgroup. Then (G, K) is a so-called Riemannian symmetric pair. G/K is equipped with a canonical G-invariant measure (see [15] for details).

As in the previous section, we shall connect the Pompeiu property with certain representations of the group G. We need only unitary irreducible representations Π of G on finite-dimensional Hilbert spaces $\mathscr{H}$. Such a representation Π is said to be a class-1 representation with respect to K if there exists a vector $0 \neq v \in \mathscr{H}$, such that for all $k \in K, \Pi(k)v = v$. For (G, K) a Riemannian symmetric pair it is well known that an irreducible representation Π is either of class-1 (in which case v is unique upto scalar multiples) or Π does not admit any nonzero vector v such that $\Pi(k)v = v$ for all $k \in K$ (see [16], p. 412). We record two simple observations for a continuous unitary irreducible representation Π of G: For $f \in L^1(G)$, we denote by $\Pi(f)$ the operator $\Pi(f) = \int_G \Pi(x) f(x)\, dx$. So if f is either right or left K-invariant, then $\Pi(f) = 0$ provided Π is not of class-1. If Π is class-1 and $0 \neq v_0$ is a K-fixed vector for Π and f is right K-invariant, then $\Pi(f) = 0$ on $(\mathbf{C} v_0)^{\perp}$, i.e., $\Pi(f)$ is completely determined by its action on v_0. On the other hand, if f is left K-invariant, $\Pi(f)$ maps $\mathscr{H}$ into $\mathbf{C}\,v_0$. If, further, f is also K-bi-invariant, then $\Pi(f)v_0 = c_{\Pi,f} \cdot v_0$ where the constant $c_{\Pi,f}$ can be explicitly computed in terms of the so-called elementary spherical functions. In fact, fix v_0 such that $\| v_0 \| = 1$ and let

$$\phi_\Pi(x) = <\Pi(x)v_0, v_0> .$$

Then ϕ_Π is continuous, in fact, a real analytic K-bi-invariant function, called the elementary spherical function associated with Π and now we have $c_{\Pi,f} = \int_G \phi_\Pi(x) f(x)\,dx$.

The following simple result, which we view as the analogue of Theorems 3.1 and 5.2 for Euclidean and non-compact spaces respectively, is implicit in the work of Berenstein and Zalcman ([9]), though not stated in this form. For sake of completeness we give a sketch of the proof.

THEOREM 6.1. *Let* $X = G/K$ *be a symmetric space of the compact type where* G *and* K *are as above. A measurable subset* $E \subseteq X$ *of positive measure has the Pompeiu property (for the usual G-action on X) if and only if for all irreducible representation* Π *of class 1 of the group* G, $\Pi\,(1_{\tilde{E}}) \neq 0$ *(where* $\tilde{E} = \{g : gK \in E\}$).

Sketch of Proof: Suppose that E fails to have the Pompeiu property. So there exists $f \in C(X)$, $f \not\equiv 0$ such that $\int_{gE} f(x)\,dx = 0$ for all $g \in G$. As in Section 5, this means $\tilde{f} * \check{1}_{\tilde{E}} = 0$. Now $\tilde{f}$ is a nonzero right K-invariant function and let Π be an irreducible representation of G such that $\Pi(\tilde{f}) \neq 0$. By an easy Peter-Weyl argument such a Π exists. By our previous discussion then Π is a class-1 representation. We have

$$\Pi(\tilde{f} * \check{1}_{\tilde{E}}) = \Pi(\tilde{f}) \cdot \Pi(\check{1}_{\tilde{E}}) = 0 \tag{6.1}$$

We claim that $\Pi(\check{1}_{\tilde{E}}) = 0$. If not, since $\check{1}_{\tilde{E}}$ is left-invariant, by our earlier discussion again $\Pi(\check{1}_{\tilde{E}})\,\mathscr{H} = \mathbf{C} \cdot v_0$ where v_0 is a nonzero K-fixed vector. But then by equation (6.1), $\Pi(\tilde{f})v_0 = 0$ and hence $\Pi(\tilde{f}) = 0$ as $\tilde{f}$ is right K-invariant, which is a contradiction. Thus $\Pi(\check{1}_{\tilde{E}}) = 0$. If now $\tilde{\Pi}(g) = \Pi'(g^{-1})$, i.e., the representation contragredient to Π, then $\tilde{\Pi}$ is again an irreducible unitary representation of class-1. We have $\tilde{\Pi}(1_{\tilde{E}}) = 0$ as desired. The converse assertion is in fact easier and follows from the Peter-Weyl theorem. This result is related to an analogue of the Wiener-Tauberian theorem for compact symmetric spaces ([30]).

In the special case when E is invariant under the left action of K as well the condition of the Theorem 6.1 reduces to a condition on the *spherical Fourier transform* of $1_{\tilde{E}}$. In fact, it follows from our earlier discussion (since

$1_{\tilde{E}}$ is K-bi-invariant) that for Π irreducible of class-1, $\Pi(1_{\tilde{E}}) = 0$ if and only if $<\Pi(1_{\tilde{E}})v_0, v_0> = 0$ where v_0 is a nonzero K-fixed vector. In terms of the elementary spherical function introduced earlier this is equivalent to saying

$$\int_G 1_{\tilde{E}}(g)\phi_\Pi(g)dg = 0 .$$

Thus using earlier notation this is the same as saying $c_{\Pi, 1_{\tilde{E}}} = 0$. Finally, for a left K-invariant set $E \subseteq X$, the condition of Theorem 6.1 is equivalent to demanding $c_{\Pi, 1_{\tilde{E}}} \neq 0$ for all Π of class-1.

As an application we consider the case when E is the geodesic ball B_r of radius r about the identity coset $eK \in X$. We work out the case of $S^2 = SO(3, \mathbf{R})/SO(2, \mathbf{R})$. The irreducible representations of class-1 of $SO(3, \mathbf{R})$ come from the decomposition

$$L^2(S^2) = \bigoplus_{k=0}^{\infty} \mathscr{H}_k$$

where $\mathscr{H}_k$ is the space of spherical harmonics of (homogeneous) degree k. On each $\mathscr{H}_k$ we have the irreducible representation

$$\Pi_k(g)f(x) = f(g^{-1}x) \quad f \in \mathscr{H}_k, \quad g \in SO(3, \mathbf{R}), \quad x \in S^2 .$$

We identify $(1, 0, 0) \in S^2$ with the identity coset $SO(2, \mathbf{R})$. The unique $SO(2, \mathbf{R})$-fixed function in $\mathscr{H}_k$ is given by $v_k(x_1, x_2, x_3) = P_k(x_1)$ where P_k is the Legendre polynomial of degree k (see [16], p. 404). Integrating first on the parallels orthogonal to $(1, 0, 0)$ and then with respect to θ where $x_1 = \cos\theta$, we get

$$c_{\Pi_k, 1_{\tilde{B}_r}} = \int_0^r P_k(\cos\theta)\sin\theta \, d\theta = P_{k+1}(r)$$

by the properties of Legendre polynomials.

Summarising this discussion, we get the following particular case of the general result of Berenstein and Zalcman ([9], Theorem 4) — where the computation is shown for a general rank 1 symmetric space of compact type.

PROPOSITION 6.2 (Berenstein and Zalcman). *A geodesic ball of radius* r *on* S^2 *has the Pompeiu property if and only if* $P_{k+1}(r) \neq 0$ *for* $k = 0, 1, 2, \cdots$.

See also the discussion in the section on compact groups in [24].