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LINK SIGNATURE, GOERITZ MATRICES AND POLYNOMIAL INVARIANTS

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ABSTRACT

Lickorish and Millett introduced the notion of skein equivalence of oriented links in [15]. In the first part of this paper I show that the link signature function $\sigma_L: S^1 \to Z$ of [22], [12], etc. is a skein invariant for links with non-zero Alexander polynomial. In the second part I show that a renormalised form of Kauffman's polynomial invariant $F_L(a, z)$, well-defined on non-oriented links is calculable from the Goeritz matrix.

1. *P*-SKEINS AND SIGNATURE

In this section I present two notions of "skein equivalence" for links; "broad" skein equivalence and "narrow" skein equivalence. Narrow skein equivalence is a stronger relation (i.e. has smaller equivalence classes), but it is not clear whether it is strictly stronger. I show that the link signature function $\sigma_L: S^1 \to Z$ is a broad skein invariant for all links with non-zero Alexander polynomial. It is not known whether this result extends to links with zero Alexander polynomial, but it seems unlikely that it should.

1.1. PRELIMINARIES

I briefly recap on some standard material. See [19], [2] or [5] for further details. Let L be an oriented link. Then it is always possible to find an oriented surface F embedded in S^3 in such a way that $\partial F = L$, with the appropriate orientation. Such a surface is called a *Seifert surface* for the link L. Now let $c_1, \ldots, c_n$ be closed curves lying in S whose homology classes generate $H_1(S)$, and let $c_1^+, \ldots, c_n^+$ be the results of pushing these curves slightly away from S in the positive direction in a collar neighbourhood of the surface. The matrix $V = (v_{ij})$, where $v_{ij} = lk(a_i, a_j^+)$, is called a *Seifert matrix* for the link L. Of course such a matrix is not well-defined, but it is true that any two Seifert matrices for a link L are *S-equivalent*; that is, can be transformed into each other by a finite sequence of the moves

$$V \to PVP' , \tag{1}$$

where P is an integral unimodular matrix, and

$$V \to \begin{pmatrix} V & & \\ & 0 & 1 \\ & 0 & 0 \end{pmatrix} . \tag{2}$$

Up to sign and multiplication by a power of t, the Alexander polynomial can be obtained by

$$\Delta_L(t) \doteq \det(tV - V') . \tag{3}$$

Finally, the *determinant* of L is given by

$$\det(L) = i^n \det(V + V') \tag{4}$$

where V is an $n \times n$ square matrix, and the classical *signature* of the link L is defined by

$$\sigma_L = \sigma(V + V') \tag{5}$$

and this turns out to be a well-defined link invariant.

Tristram [22], Levine [12] and others have developed the following generalisation of the classical signature of a link: Let V be a Seifert matrix for a link L, and ω a complex number of modulus 1. Instead of considering the symmetric matrix $V + V'$, we can deal with the Hermitian matrix $H(\omega) = (1-\bar{\omega})V + (1-\omega)V'$. The fact that $|\omega| = 1$ enables us easily to show that the signature $\sigma(H(\omega))$ is unchanged by the moves (1) and (2) and so takes the same values for S-equivalent matrices. It hence provides a function $\sigma_L: S^1 \to Z$ which may be regarded as a link invariant. Further, $H(\omega) = (\omega - 1)(\bar{\omega}V - V')$ so $\sigma_L(\omega)$ is continuous away from roots of the Alexander polynomial. Clearly $\sigma_L(-1)$ is twice the classical signature of a link.

By a *skein triplet* (or *oriented skein triplet*) of links I shall mean a triplet (L_+, L_-, L_0) of oriented links which are identical outside some ball $B \subset S^3$ and inside it are as shown in Figure 1.

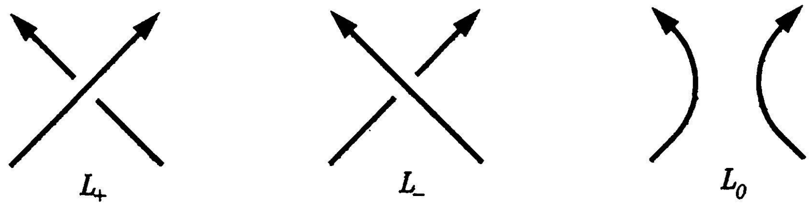

FIGURE 1

There are two equivalence relations with which we may be interested: Let R_b be the equivalence relation on the set of links generated by:

$$\text{(6)}\quad \text{if } (L_+, L_-, L_0) \text{ and } (L'_+, L'_-, L'_0) \text{ are skein triples then } \begin{cases} L_+R_bL'_+, L_-R_bL'_- \Rightarrow L_0R_bL'_0 \\ L_+R_bL'_+, L_0R_bL'_0 \Rightarrow L_-R_bL'_- \\ L_-R_bL'_-, L_0R_bL'_0 \Rightarrow L_+R_bL'_+ . \end{cases}$$

I shall call this equivalence relation *broad oriented skein equivalence*. The other relation on the set of oriented links, *narrow oriented skein equivalence*, is the equivalence relation R_n generated by:

$$\text{(7)}\quad \text{if } (L_+, L_-, L_0) \text{ and } (L'_+, L'_-, L'_0) \text{ are skein triples then } \begin{cases} L_+R_nL'_+, L_0R_nL'_0 \Rightarrow L_-R_nL'_- \\ L_-R_nL'_-, L_0R_nL'_0 \Rightarrow L_+R_nL'_+ \end{cases}.$$

It is obvious that R_b is a weaker equivalence relation than R_n (i.e. the equivalence classes are larger), but it is not clear (and I do not know) whether it is strictly weaker. By the *broad* or *narrow oriented skein of links* I refer to the set of equivalence classes of oriented links under the relation R_b or R_n (Note that in most of the literature, R_n is referred to as "skein equivalence"; R_b is not referred to at all). The polynomial invariant $P_L(l, m)$ of [15], [3] etc. may be regarded as the most general linear broad skein invariant (see [15], [16]). The fact that the value of $P_L(l, m)$ specified on the unknot U is sufficient to define its value on any link may be taken as saying that the broad oriented skein is *generated* by U. The corresponding statement for the narrow oriented skein is that specifying the values of $P_L(l, m)$ on all unlinks is sufficient to define its values on all links — the set of unlinks generates the narrow oriented skein.

1.2. SIGNATURE AND ORIENTED SKEINS

I now show that the signature function $\sigma_L(\xi)$ of any link with non-zero Alexander polynomial is a broad oriented skein invariant (It is already known that the signature $\sigma = \frac{1}{2}\sigma_L(-1)$ is a narrow skein oriented invariant