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**Autor:** Schrauwen, Robert

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spectrum determines the topology of a plane curve singularity) and the — equivalent — conjecture involving the real Seifert form (cf. [SSS]). Also, A. Neméthi used the idea of topological series to define his topological trivial series [Nm].

In the Appendix, we have included the EN-diagrams and some invariants of the Arnol'd series.

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## 2. Splicing and series

2.1. It is clear that singularities occur in series. The simplest series have been given names, such as *A, D, J,* etc., by Arnol'd. But how to define a series is unclear. One looked at deformation properties such as adjacencies, etc., because the goal is to define what a series means analytically. A proper analytical description can be given for series of the form $f + l^k$, where $l$ is a sufficiently general linear form, see the work of Iomdin and Lê, [Lê]. But already in the case of Arnol'd's series, one finds that they are not of the 'Iomdin-type'. Some series are multi-indexed, such as

$$Y_{r,s}\colon x^2y^2 + x^{r+4} + y^{s+4} \,,$$

and others, such as $W^{\#}$:

$$\begin{aligned} W^{\#}_{1,2q-1}\colon &\quad (y^2-x^3)^2 + x^{4+q}y \\ W^{\#}_{1,2q}\colon &\quad (y^2-x^3)^2 + x^{3+q}y^2 \end{aligned}$$

make smaller steps than a linear series.

However, the most apparent properties that hold a series together, are the topological invariants. For example, the Milnor number within Arnol'd's series, increases with steps of 1. Therefore it is worthwhile to go not as far as an analytical definition, but to look for a topological one.

Another property is that, as already mentioned in the Introduction, series of isolated singularities are clearly related to non-isolated singularities, and that the hierarchy of these non-isolated singularities reflects the hierarchy of the isolated singularities. This relationship is also not completely understood. Our topological definition, which works for plane curve singularities, makes clear which isolated singularities belong to the series of a given non-isolated singularity.

2.2. The motivation for our definition comes from the topology of the link exterior. We first need to recall some facts of splicing and EN-diagrams.

Let $f \in \mathbf{C}\{x, y\}$ be a plane curve singularity, and $L$ the link of $f$ embedded in $S^3$. $L$ completely describes the topological type of $f$. There is a notation for $L$ by means of a weighted graph, that we call an EN-diagram, introduced by Eisenbud and Neumann [EN]. The EN-diagram of $f$ is closely related to the resolution graph of $f$ (the dual graph of the good minimal resolution). In fact, as a graph, the EN-diagram is equal to the resolution graph with all linear chains contracted. We will call the vertices of valence 1 *dots,* and the vertices of valence at least 3 *nodes.* The arrows correspond to the components of $L$ (or the irreducible components of $f$), and they have a multiplicity, equal to the multilink multiplicity. The nodes, dots and edges have topological meanings as well, we refer to [EN] for details. There are conversion rules from EN-diagram to resolution graph and back, see [EN], chapter V.

It is known that $M = S^3 \setminus N(L)$ (where $N(L)$ is a open tubular neighbourhood of $L$) is a *Waldhausen manifold,* see [EN] or [LMW]. This means that there is a decomposition of $M$ in Seifert manifolds (the basic building blocks). The decomposition can be found in several ways, e.g. by means of the resolution of $f$ or the polar decomposition of $f$. This is explained in detail in [LMW].

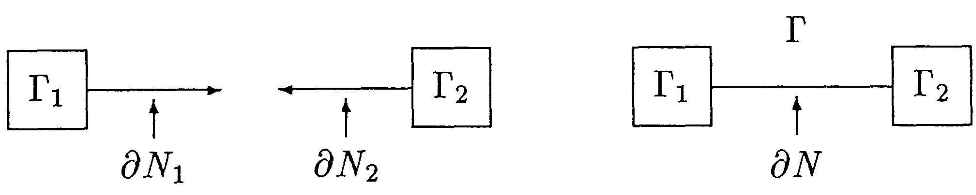
2.3. Glueing two pieces of this decomposition together uses the operation of *splicing,* due to L. Siebenmann and studied extensively in [EN]. Consider two (multi)links $L_1 = m_1 S_1 + L_1'$, $L_2 = m_2 S_2 + L_2'$, embedded in (separate copies of) $S^3$. Let $N_1, N_2$ be small tubular neighbourhoods of $S_1, S_2$. Then the *splice* $L$ of $L_1$ and $L_2$ is the link

$$L = L_1' + L_2' ,$$

embedded in the homology sphere

$$\Sigma = (S^3 \setminus N_1) \cup_{\partial} (S^3 \setminus N_2) ,$$

the boundaries $\partial N_1$ and $\partial N_2$ of the tubular neighbourhoods glued meridian to longitude and vice versa. The EN-diagram $\Gamma$ of $L$ arises from the EN-diagrams of $L_1$ and $L_2$ by replacing the two arrows representing $S_1$ and $S_2$ by an edge (which represents the splice torus $\partial N_1 = \partial N_2$):



If we impose two conditions, described below, then $L$ is again an algebraic link in $S^3$.

SPLICE CONDITION

$$m_1 = \mathrm{lk}(S_2, L_2') \quad \text{and} \quad m_2 = \mathrm{lk}(S_1, L_1')\ ,$$

i.e. $m_1$ has to be equal to the linking number of $S_2$ with the other components of $L_2$, counted with their multiplicities (and similarly for $m_2$).

Linking numbers can be computed easily from the EN-diagram, see [EN], section 10.

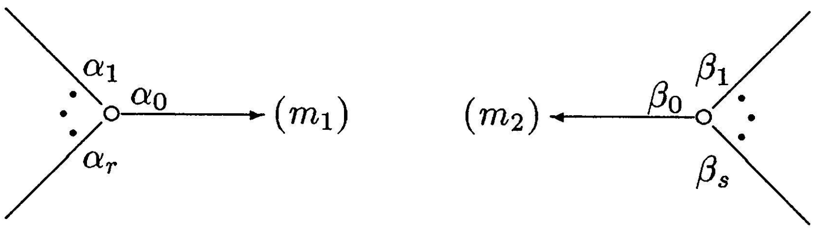
If the splice condition holds, then $L$ is again a fibred link. In fact it forces that the Milnor fibres cut the splice torus in an $(m_1, m_2)$-torus link.

The second condition is a condition on the weights of the EN-diagram. It follows from [EN], Theorem 9.4, that we need the following condition in order that $L$ is again algebraic:

ALGEBRAICITY CONDITION

(a) The resulting link can be obtained by repeated cabling, and

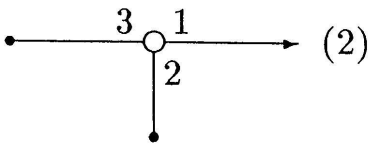
(b) If the EN-diagrams of both links near the splice arrows are as follows:




then the inequality $\alpha_0\beta_0 > \alpha_1 \cdots \alpha_r\beta_1 \cdots \beta_s$ must hold.

2.4. We now return to the ideas behind our definition of series. A typical series is the series consisting of $W^{\#}_{1,2q-1}$ and $W^{\#}_{1,2q}$, introduced earlier. Their EN-diagrams are:


It is clear that this is the result of splicing something to



which is precisely the EN-diagram of $W^{\#}_{1,\infty}$: $(y^2-x^3)^2$. In terms of the resolution: take a resolution of $f(x, y) = (y^2-x^3)^2$, and deform the double component slightly into an $A_p$, we then get a — partial — resolution of one of the $W^{\#}_{1,p}$. In terms of the splice decomposition: Consider the splice decomposition of a representative $f_p$ of $W^{\#}_{1,p}$. It consists of two pieces, one of which is the complement of the link of $f$, wheras the other depends on the parameter $p$. This is equivalent to the statement that the Milnor fibration of $f_p$ results from the Milnor fibration of $f$ by removing a tubular neighbourhood of the link component, and replacing it by something else in such a way that the result is the Milnor fibration of $f_p$ and leaving the rest unchanged. We will see later that this process will not give more than only the series $W^{\#}_{1,p}$. The link of $f_p$ is a $(2, 6+p)$-cable on the link of the reduced singularity $f_R(x, y) = y^2 - x^3$, which is a $(2, 3)$-torus knot.

If we have a singularity with more than one double component, we can splice something to each of the components independently. We see this with our example $Y_{r,s}$, its EN-diagram is the result of splicing two pieces (one depending on $r$ and one on $s$) to $(2) \leftrightarrow (2)$, the EN-diagram of $Y_{\infty,\infty}$: $x^2y^2$.

If we have a singularity $f$ with a component of multiplicity greater than two, then we can get non-isolated singularities with lower multiplicities when we splice something to it. The simplest example is $f(x, y) = y^3$. In this case, the Milnor fibre consists of three discs. If we want to replace a small tubular neighbourhood of the knot with something else, in such a way that the result is again an algebraic link, we first of all have to take care that the fibres in the solid torus that we put back in, approach the boundary in a $(3, 0)$-torus link. It is intuitively clear that this is only possible with 3 components of multiplicity 1 or with 1 single and 1 double component. Indeed, in 3.8 we will see, that this gives the possibilities $E_{6k}$, $E_{6k+1}$, $E_{6k+2}$ and $J_{k,\infty}$, and if we apply the same procedure again to $J_{k,\infty}$, we get the series $J_{k,p}$. In Arnol'd's list we find all these singularities in the series of $y^3$.

In the Appendix we have included the EN-diagrams of all Arnol'd series, and one sees that they all arise from splicing something to the link of the corresponding non-isolated singularity.

These examples motivate our definition of topological series, which will be presented next.