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Autor: Schrauwen, Robert

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3. THE DEFINITION OF TOPOLOGICAL SERIES

3.1. *Definition.* Let $f \in \mathcal{O} = \mathbf{C}\{x, y\}$ have a non-isolated singularity. The *topological series* belonging to f consists of all topological types of isolated singularities whose link arise as the splice of the link of f with some other link.

So what we want is that the Milnor fibration of an element of the series differs from that of f only in small neighbourhoods of the components with higher multiplicities.

In terms of EN-diagrams: All arrows in the diagram of f with '(m)' $(m > 1)$ in front of them, have to be replaced by subdiagrams with arrows with multiplicity 1 only, taking the splice and algebraicity conditions into consideration. The advantage of using EN-diagrams instead of resolution graphs can be observed here: it is not easy to describe the linear chains that arise in the resolution graphs of the isolated singularity.

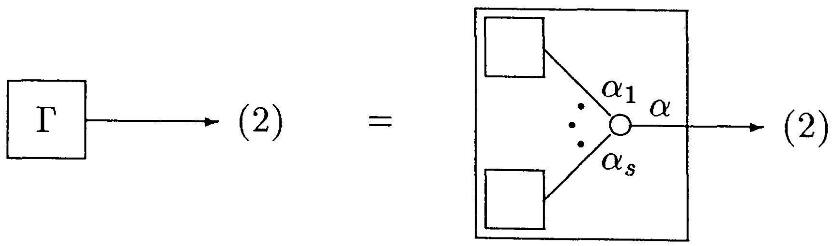
Below, we investigate what possibilities there are to replace an arrow '$\rightarrow (m)$' by something else, in the sense of the preceding remarks. It will follow that the topological series do not contain more singularities than we want them to. The method is purely combinatorial. We start with $m = 2$ and end with a formula giving the number of such possibilities.

3.2. *Notation.* If Γ is an EN-diagram, then we denote by $A(\Gamma)$ the set of arrow-heads of Γ, by $N(\Gamma)$ the set of non-arrow-heads (dots and nodes) and by $V(\Gamma) = A(\Gamma) \cup N(\Gamma)$ the set of all vertices.

The corresponding (multi)link is $L = L(\Gamma) = \sum_{i \in A(\Gamma)} m_i S_i$, and for $i \in N(\Gamma)$, S_i will denote the corresponding *virtual* component (cf. [EN]).

3.3. THE CASE OF A DOUBLE COMPONENT.

Suppose $f \in \mathcal{O}$ has link $L = \sum_{i \in A(\Gamma)} m_i S_i$. Suppose one of the components, $S_\diamond$, has multiplicity 2, i.e. $m_\diamond = 2$. Near the arrow $\diamond$, the EN-diagram Γ of L looks like this:



where the boxes may denote anything and the arrow is $\diamond \in A(\Gamma)$ (the second picture is only defined when $\Gamma \neq \bullet\!\rightarrow (2)$). Define the following numbers:

$$N_0 = \left[\frac{2\alpha_1 \cdots \alpha_s}{\alpha}\right] \text{ , where [.] denotes integral part ,}$$

$$c = \sum_{j \in A(\Gamma), j \neq \diamond} m_j \operatorname{lk}(S_\diamond, S_j) \text{ ,}$$

i.e. c is the linking number of $S_\diamond$ with the other components, counted with their multiplicities.

Note that we work with *minimal* EN-diagrams, which means that redundant dots (those attached to a node with weight 1) must be removed by using theorem 8.1 of [EN].

We now show what possibilities there are to replace the double component, in the sense of the remarks at the beginning of this section. Let

$$\Delta_k = \det(tI - h_*)$$

be the characteristic polynomial of the monodromy on $H_k(F)$, and let $\Delta_* = \Delta_1/\Delta_0$. This function is related to the zeta function ζ_f of the monodromy (cf. [A'C]) by the relation $\zeta_f(f) = t^{-\chi(F)}\Delta_*(t^{-1})$ (where $\chi(F)$ is the Euler characteristic of F).

3.4. THEOREM. *The only two (classes of) possibilities to replace a double component, are:*

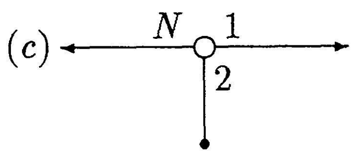
with $N > N_0$ odd   with $N > N_0$ even

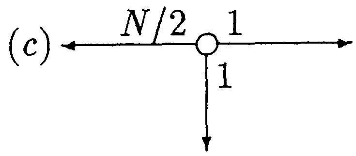
Furthermore, let Δ_*^∞ *be the* Δ_* *of* $L = L(\Gamma)$, *and* Δ_*^N *be the* Δ_* *of the new link. Then we have:*

$$\Delta_*^N(t) = \Delta_*^\infty(t) \cdot \left(t^{N+c} - (-1)^N\right)$$

In particular, the Milnor number is linear in N *with coefficient one.*

Proof. The EN-diagrams of the theorem can be regarded as being the results of splicing the links $L = L(\Gamma) = L(f)$ and those defined by the EN-diagrams Γ'_N in the next figure, along the components $S_\diamond$ and the one with multiplicity c, which we call S'_* with $* \in A(\Gamma'_N)$. (Note that c can be zero, in [EN] this has been given a natural interpretation).





That the multiplicity must be c follows from one half of the splice condition. The other half, $2 = \sum_{h \in A(\Gamma'_N), h \neq *} m_h \operatorname{lk}(S'_*, S'_h)$, implies that these two diagrams are the only two essentially different EN-diagrams with the required property, for we want $m_h = 1$. For the first link the splice condition reads '$2 = 2 \cdot 1$' and for the second '$2 = 1 \cdot 1 + 1 \cdot 1$'.

Finally, the algebraicity condition gives $N > N_0$.

The Δ_* formula follows from [EN], theorem 4.3. □

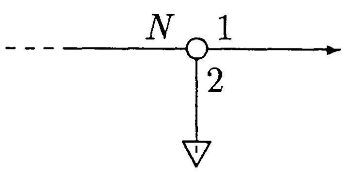
The last statement of the theorem implies that if L is not the unknot, the Milnor numbers are related as follows:

$$\mu_N = \mu_\infty + N + c \qquad \text{if} \quad F \text{ is connected},$$

$$\mu_N = \mu_\infty + N + c - 1 \quad \text{if} \quad F \text{ is not connected}.$$

We see for example that in case f is of type A_∞, the series is precisely the whole A-series, and in case $f(x, y) = x^2y^2$, the series is the complete doubly indexed Y-series.

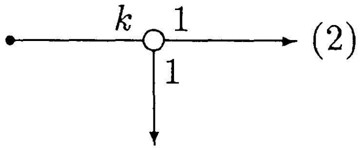
3.5. *Definition.* We combine the two possibilities in one graph, where, depending on whether N is odd or even, the first or the second graph of the theorem must be substituted.



Observe that for N even, this represents the graph with two arrows and edge weight $N/2$.

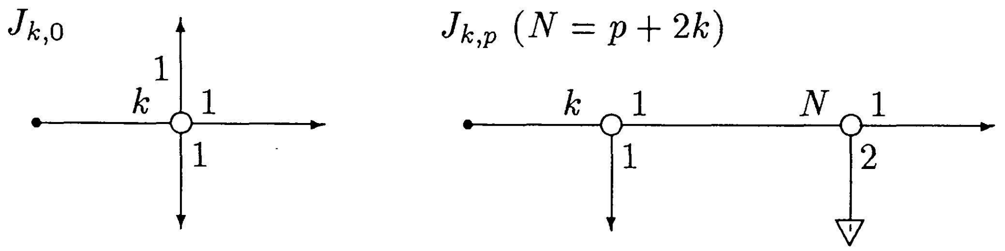
3.6. *Remark.* If $\alpha = 1$ or $\alpha = 2$ (see the figure at the beginning of this section), then the case $N = N_0$ is also allowed, although then the diagram has to be minimized by applying theorem 8.1 of [EN]. The monodromy formula still holds.

3.7. *Example.* $J_{k,\infty}$ has the equation $f(x, y) = y^2(y + x^k)$, its EN-diagram is pictured below:



$$\Delta_1^\infty(t) = (t-1)\frac{t^{3k}-1}{t^3-1}$$

We have $c = k$ and $N_0 = 2k$. The series is $J_{k,p}: y^3 + y^2x^k + x^{3k+p}, p \geqslant 0$. The case $p = 0$ is the special case with $N = N_0$.



We have

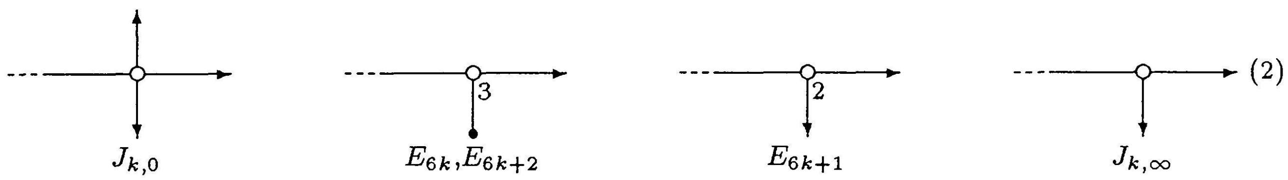
$$\Delta^N(t) = (t-1)\left(t^{N+k} - (-1)^N\right)\frac{t^{3k}-1}{t^3-1}.$$

3.8. HIGHER MULTIPLICITIES.

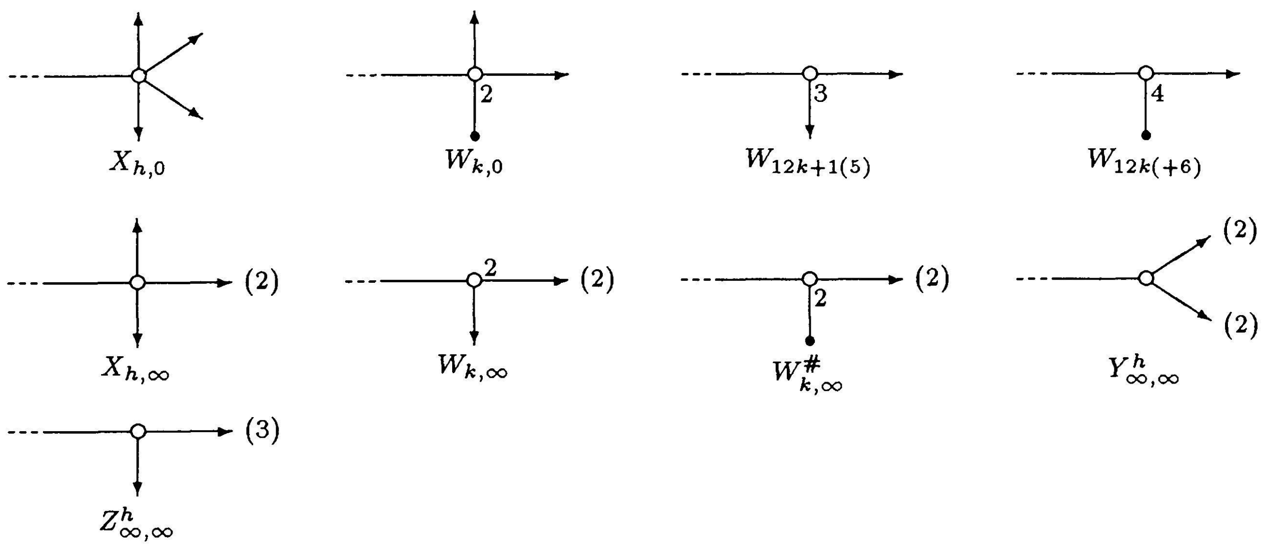
When we have higher multiplicities, exactly the same method can be used. The splice condition gives us always a finite number of links that can be spliced to the component with multiplicity m. We enumerate the possibilities when $m = 3$ and $m = 4$. The names refer to the simplest case when $f(x, y) = y^m$.

In the diagrams, the splice edges have variable weight N, N having no common factor with the other weights. Further omitted edge weights are equal to 1. We only listed the diagrams with one node; some have an arrow of multiplicity greater than 1, which should be treated again.

The four possibilities for $m = 3$:



The nine possibilities for $m = 4$:



We now give a formula giving the number of essentially different diagrams with one node and only multiplicities less than m, that can be spliced to a component of multiplicity m.

PROPOSITION. *The number is:*

$$\sum_{q|m} \mathfrak{p}(m/q) + \sum_{1 \leqslant p \leqslant m-1} \; \sum_{q|(m-p),\, q>1} \mathfrak{p}((m-p)/q) - 1$$

where $\mathfrak{p}(n)$ *is the number of integer partitions of* n.

Proof. In such a diagram at most one dot appears, with at the node a weight $\geqslant 2$. The number of edges emerging from the node must be at least 3. There is at most one weight $\geqslant 1$. These are consequences of the algebraicity condition. The splice condition demands that the total linking number of the other components with the splice component equals m. The formula is now a matter of counting. □

For $m \leqslant 15$ we obtain:

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
number	0	2	4	9	12	22	27	42	54	76	91	134	159	211	263

This can be regarded as an upperbound on the number of symbols (such as A, $W^{\#}$, etc.) needed to give names to all singularities of corank m.

4. THE SPECTRUM OF A PLANE CURVE SINGULARITY

4.1. In this section we compute the spectrum of a plane curve singularity from the EN-diagram and we prove a splice formula for spectra. This will be needed in the next section, where we look at several invariants within a series. First we need to define a number of polynomials.

4.2. We denote by F the Milnor fibre of a plane curve singularity f.

Definition.

$$\Delta_0(t) = \text{char. pol. of } H_0(h): H_0(F) \to H_0(F)\,,$$
$$\Delta_1(t) = \text{char. pol. of } H_1(h): H_1(F) \to H_1(F)\,,$$
$$\Delta_*(t) = \Delta_1(t)/\Delta_0(t) \in \mathbf{Q}(t)$$