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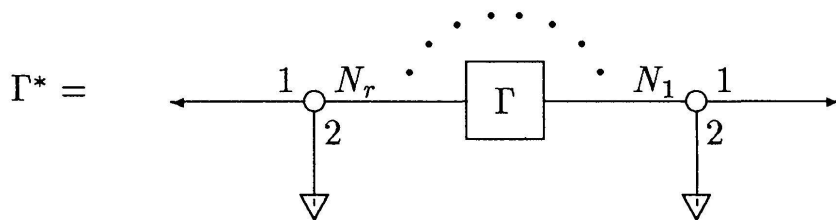
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*Proof.* If  $q = 1$  the theorem is clear. Now suppose  $q > 1$ . Consider the behaviour of the polynomials  $\Delta_0, \Delta^1$  and  $\Delta'$  under this splice operation. Splicing introduces a new edge  $E$  which contributes to  $\Delta^1$  with a factor  $t^q - 1$ . This introduces new  $2 \times 2$ -Jordan blocks. Both splice components have  $\sum_{i=1}^{q-1} \left( -\frac{i}{q} \right)$  in their spectrum (coming from  $\Delta'$ ). But, as both eigenvalues in a  $2 \times 2$ -block are of different weight,  $L$  has  $\sum_{i=1}^{q-1} \left( -\frac{i}{q} \right) + \left( \frac{i}{q} \right)$  instead of the sum of both parts. It is clear from theorem 4.8 that all other parts of the spectra of  $L'$  and  $L''$  have to be added.  $\square$

5. INVARIANTS IN THE CASE  
 THAT  $f$  HAS ONLY TRANSVERSAL  $A_1$  SINGULARITIES

In this section we describe the topology and equation of a topological series that belongs to a non-isolated singularity with only transversal  $A_1$  singularities.

Throughout this section,  $f \in \mathcal{O}$  is of the form  $f = f_1^2 \cdots f_r^2 g$ , with  $f_1, \dots, f_r$  irreducible and  $g$  reduced. The critical set of  $f$  is  $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$ , and the transverse type of  $f$  along  $\Sigma_i$  is  $A_1$ . For all  $i \in \{1, \dots, r\}$ , we have numbers  $N_{0i}$  and  $c_i$  as defined in section 3.3. Let  $N_i > N_{0i}$  ( $1 \leq i \leq r$ ). According to theorem 3.4, a typical element of the series belonging to  $f$  has the topological type (EN-diagram)  $\Gamma^*$ :



That is: each arrow of the EN-diagram  $\Gamma$  of  $f$  belonging to a double component, is replaced in the way described in theorem 3.4. So varying the  $N_i$  will give us the complete series belonging to  $f$ .

The following two propositions are easy consequences of theorem 3.4. Let  $N = (N_1, \dots, N_r)$  and let  $f_N$  have topological type  $\Gamma^*$ .

5.1. PROPOSITION. Let  $\Delta_*[f]$  and  $\Delta_*[f_N]$  be the  $\Delta_*$  of  $f$  and  $f_N$  respectively. Then:

$$\Delta_*[f_N](t) = \Delta_*[f](t) \cdot \prod_{i=1}^r (t^{N_i+c_i} - (-1)^{N_i}). \quad \square$$

5.2. PROPOSITION. Let  $\mu_\infty$  be the Milnor number of  $f$  and  $\mu_N$  that of  $f_N$ . Let  $\mu_0 \in \{1, 2\}$  be the number of connected components of the Milnor fibre of  $f$ . Then:

$$\mu_N = \mu_\infty - \mu_0 + 1 + \sum_{i=1}^r (N_i + c_i).$$

The numbers  $\mu_\infty, \mu_0$  and  $c_i$  ( $1 \leq i \leq r$ ) depend only on  $f$ . □

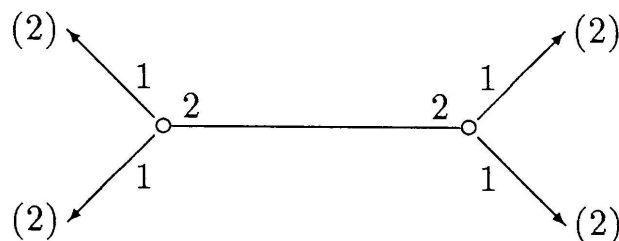
5.3. We conclude this list of topological invariants with the formula of the spectrum of a series (see section 4).

PROPOSITION. Define for  $1 \leq i \leq r$ :  $\gamma_i = 0$  if  $c_i$  is even and  $\gamma_i = 1/2$  when  $c_i$  is odd. Write  $v_i = N_i + c_i$ . Then:

$$\text{Sp}(f_N) = \text{Sp}(f) + \sum_{i=1}^r \sum_{j=0}^{v_i-1} \left( \frac{1}{2} - \frac{\gamma_i + j}{v_i} \right)$$

*Proof.* One can use the proof of [St], theorem 4.5, but it is also possible to work out the various cases using the method of section 4. For the proof of [St], the following observation is needed. Let  $F'_i$  be a transversal slice transverse to  $\Sigma_i$  (in this case  $F'_i$  consists of two points — the transverse type is  $A_1$ ). Let  $T_i: H_0(F'_i) \rightarrow H_0(F'_i)$  be the monodromy of the local system over the punctured disc  $\Sigma_i \setminus \{0\}$ . Then it is well-known that  $T_i$  is the identity if  $c_i$  is even and  $-$  identity if  $c_i$  is odd. In fact, even if the transverse type is not  $A_1$ , the following holds. Let  $t_i: H_0(F'_i) \rightarrow H_0(F'_i)$  be the Milnor fibration monodromy of  $f$  restricted to a transversal slice through  $x \in \Sigma_i$ . Then  $t_i$  is a cyclic permutation of the finite number of points in  $F'_i$ , and  $T_i = t_i^{-c_i}$ . □

*Example.* Let  $f(x, y) = (y^2 - x^4)^2(x^2 - y^4)^2$ ; its EN-diagram is:



Observe that according to the proposition, the spectrum of  $f_N$  is independent of the order of  $N_1, \dots, N_4$ . If we take  $N = (5, 5, 6, 6)$  and  $N = (5, 6, 5, 6)$  we get the same spectrum but different topological types, because the EN-

diagrams are not equivalent. This is the counterexample to the spectrum conjecture found by Steenbrink and Stevens, cf. [SSS].

### 6. EQUATIONS

In this section we discuss the equations of series: what do we have to add to  $f$  to obtain a required element of its series?

In the example  $W^\#$  at the beginning of section 4, we had:

$$W_{1,2q-1}^\# : (y^2 - x^3)^2 + x^{4+q}y$$

The Puiseux expansion of  $W_{1,\infty}^\# : f(x, y) = (y^2 - x^3)^2$ , is  $x = t^2, y = t^3$ . When we substitute this in  $x^{4+q}y$ , we get  $t^{11+2q}$ , which is just the number  $N$  in the EN-diagram.

More generally, it appears that adding  $\varphi \in \mathcal{O}$  with  $\varphi(t^2, t^3)$  of order  $11 + 2q$ , gives the same result, although there are various kinds of exceptions.

In theorem 6.5 below, we give conditions on  $\varphi$  such that  $f + \varepsilon\varphi$  has the required type, where  $\varepsilon$  is introduced in order to fulfil transversality properties. This avoids exceptional cases such as when  $f(x, y) = y^2$  and  $\varphi(x, y) = 2x^k y + x^{2k}$ , the sum is then a non-isolated singularity.

Again,  $f$  has only transversal  $A_1$  singularities; but the following lemma is valid in greater generality.

6.1. LEMMA. *Let  $f, \psi \in \mathcal{O}$  and assume  $f$  has a non-isolated singularity. If for all small  $\varepsilon > 0$ ,  $f + \varepsilon\psi$  has a singularity topologically equivalent to  $f$ , then for almost all  $\varepsilon$  the zero sets of  $f$  and  $f + \varepsilon\psi$  are equal.*

*Proof.* First take  $f(x, y) = y^n$  with  $n > 1$ . Assume that for no  $\varepsilon$ ,  $f$  and  $f + \varepsilon\psi$  have the same zero set. Then we may assume  $f + \varepsilon\psi = (y + F(x, \varepsilon))^n$  where  $F(x, \varepsilon) \not\equiv 0$ , regarded also as a function, of  $\varepsilon$ , can be written as

$$F(x, \varepsilon) = \sum_{i>0} a_i(\varepsilon)x^i .$$

Here  $a_i(\varepsilon)$  may have positive fractional powers of  $\varepsilon$ .  $f + \varepsilon\psi$  is linear in  $\varepsilon$ . By writing out the equation