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## §2. The infinitesimal Coxeter invariant

In his paper [5] Coxeter describes the "inversive distance" between two non-intersecting circles (cf. H.G. Forder [8] and H.W. Alexander [1] for alternate treatments). Starting with the standard formula

$$d^2 = a^2 + b^2 - 2ab\cos\theta$$

relating the sides and angle of a Euclidean triangle one obtains the formula:

$$\cos\theta = \frac{a^2 + b^2 - d^2}{2ab} \tag{2.1}$$

for the cosine of the angle between two intersecting circles in terms of the two radii $a$ and $b$, and the distance $d$ between the centres. Although the left hand side of 2.1 makes sense only for intersecting (or tangent) circles, the right hand side makes sense even for disjoint circles. The various cases are:

| $\mu = \frac{a^2 + b^2 - d^2}{2ab}$ | circles |
|---|---|
| $\lvert\mu\rvert < 1$ | intersecting |
| $\mu = 1$ | internal tangency |
| $\mu = -1$ | external tangency |
| $\lvert\mu\rvert > 1$ | disjoint |

Geometric Significance of $\mu$

Coxeter defines the *inversive distance* $\delta$ between *disjoint* circles by the formula $\cosh\delta = \lvert\mu\rvert$. Like the ordinary angle between two intersecting circles, inversive distance is a conformal invariant of the relative position of the two disjoint circles.

We shall apply this formula to compute the inversive distance between the osculating circles of two nearby points on a curve. Let $\gamma$ be a smooth curve in $\mathbf{C}$ parametrized by arc-length $z:(\alpha,\beta)\to\mathbf{C}$. Let $\kappa(s)$ be the ordinary Euclidean curvature of $\gamma$ at the point $z(s)$. The radius of the osculating circle is $1/\lvert\kappa\rvert$ and its centre lies at $z + iz'/\kappa$. Comparing the osculating circles at points $z(s)$ and $z(s+h)$ we find that the inversive distance between them is given by

$$\cosh \delta(s+h, s) = \frac{\dfrac{1}{\kappa(s+h)^2} + \dfrac{1}{\kappa(s)^2} - \left| z(s+h) - z(s) + i \left\{ \dfrac{z'(s+h)}{\kappa(s+h)^2} - \dfrac{z'(s)}{\kappa(s)^2} \right\} \right|}{\dfrac{2}{\kappa(s+h)\,\kappa(s)}}$$

Expanding this in a Taylor series in $h$ gives:

$$(2.2) \qquad 1 + \frac{\delta^2}{2} + \ldots = \cosh \delta = 1 + \frac{1}{4!} \kappa'(s)^2 h^4 + O(h^5) .$$

We note in particular that if $\kappa'(s) \neq 0$, the right hand side of this expression is larger than 1 for small $h$, proving that the osculating circles of nearby points on a curve with $\kappa' \neq 0$ are disjoint. It follows from this that the set of all osculating circles of such a curve forms a nested family.

From 2.2 we get

$$\sqrt{\delta(s+h, s)} = 12^{-1/4} \sqrt{|\kappa'(s)|}\; h + O(h^2) .$$

It follows immediately that $\omega_\gamma = \sqrt{|\kappa'(s)|}\, ds$ is a differential 1-form on the curve which is invariant under the action of the Möbius group in the following sense: for all inversions $\varphi$, the 1-form $\omega_\gamma$ is the pull-back by $\varphi$ of the 1-form $\omega_{\varphi(\gamma)}$ associated to the curve $\varphi(\gamma)$. We call $\omega = \omega_\gamma$ the "infinitesimal Coxeter invariant".

*Definition.* A vertex of $\gamma$ is a zero of $\omega$.

The invariance of $\omega$ means in particular that the property of being a vertex is an inversive invariant.

We define the inversive arc-length of a curve $\gamma$ to be the integral

$$\int_\gamma \omega .$$

This is an inversive invariant of $\gamma$. Indeed, fixing a point $a \in \gamma$, we can parametrize the curve by means of the natural parameter $\nu$, where

$$\nu(p) = \int_a^p \omega .$$

For example, the inversive arc-lengths for the conics can be calculated by means of the integrals

$$4\sqrt{3a(a^2-1)} \int_0^{\pi/2} \frac{\sqrt{\cos\theta \sin\theta}\, d\theta}{\cos^2\theta + a^2\sin^2\theta} \quad \text{for the ellipse} \quad x^2 + \frac{y^2}{a^2} = 1$$

and

$$4\sqrt{3a(a^2+1)} \int_0^{\pi/2} \frac{\sqrt{\cos\theta}\ d\theta}{a^2 + \cos^2\theta} \quad \text{for the hyperbola} \quad x^2 + \frac{y^2}{a^2} = 1\ .$$

Only in the case of a circle and a parabola do we get an elementary integral. Here is a table of the inversive arc-lengths of various conics.

| Type | $e$ | Inversive Length | Type | $\theta$ | Inversive Length |
|---|---|---|---|---|---|
| Circle | 0.0 | 0.0 | Parabola | 0.0 | $\sqrt{6}\ \pi = 7.6953$ |
| Ellipses | 0.1 | 0.59 | Hyperbola | 0.1 $\pi$ | 12.10 |
| | 0.2 | 1.19 | | 0.2 $\pi$ | 10.71 |
| | 0.3 | 1.80 | | 0.3 $\pi$ | 9.60 |
| | 0.4 | 2.45 | | 0.4 $\pi$ | 8.63 |
| | 0.5 | 3.15 | | 0.5 $\pi$ | 7.70 |
| | 0.6 | 3.92 | | 0.6 $\pi$ | 6.77 |
| | 0.7 | 4.81 | | 0.7 $\pi$ | 5.79 |
| | 0.8 | 5.91 | | 0.8 $\pi$ | 4.68 |
| | 0.9 | 7.48 | | 0.9 $\pi$ | 3.30 |

$e$ = eccentricity $\theta$ = angle between the asymptotes

TABLE 2.3

As the reader may show, when the eccentricity approaches 1 the inversive lengths of both the ellipses and the hyperbolas approach a common value which is *twice* the inversive length of the parabola. This fact and the above table are reminders that inversive geometry behaves quite differently than projective or affine geometry, which don't distinguish among the various ellipses for example. In particular it follows that the inversive arc-length is neither a projective nor an affine invariant.

We note also that the table gives the inversive lengths of any curve equivalent to these conics under a linear fractional transformation. For instance, inverting the parabola with respect to its focal point yields the cardioid which must therefore have the same inversive length as the parabola.