Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 36 (1990)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE INVERSIVE DIFFERENTIAL GEOMETRY OF PLANE CURVES

Autor: Cairns, G. / Sharpe, R. W.

Kapitel: §3. The four vertex theorem in R^2

DOI: https://doi.org/10.5169/seals-57907

§3. THE FOUR VERTEX THEOREM IN $\mathbf{R}^2$

Let γ be a closed embedded curve on $\mathbf{R}^2$. The Euclidean curvature κ is defined and so it must have a minimum and a maximum which give two vertices on γ. (Indeed the number of local minima must be the same as the number of local maxima, so that the number of extrema is even.) Next we move γ by a Möbius transformation so as to send one of these extrema to ∞, and in such a way that the curve becomes asymptotic to the x-axis. Now $\kappa(s) \to 0$ as $s \to \pm \infty$, and the theorem of turning tangents ([6], p. 37) says that $\int \kappa(s)\, ds = 0$. It follows that κ cannot have just one maximum or just one minimum for if so it would have a fixed sign and then the integral could not be zero. Thus γ has at least 2 extrema in addition to the one at infinity. But since the total number of extrema is even, there must be at least four of them, and hence four vertices.

There is a subtle point which we have glossed over in this argument. The vertices come in two types. As well as the extrema of κ (the "honest" vertices) there may also be non-extremal critical points of κ. The above "proof" has used the fact that not only are the vertices inversive invariants, but so too are the isolated extrema. Whereas this is indeed true (as is implied by equation 4.3 of the next section), it suffices to note that the non-extremal critical points of κ are unstable phenomena and each of them may be eliminated by a deformation of the curve with support in a small neighborhood of it. One may thus assume that all of the vertices of γ are extrema, whereupon the above proof stands as is.

Remark 3.1. The reader may compare the above proof to that of [10], where the four vertex theorem is obtained by using a Möbius transformation to send a *non*-vertex point to infinity.

§4. A GENERALIZATION OF THE INVARIANCE OF ω

Let (M, h) be a Riemannian surface with metric h, and let γ be a curve on M with geodesic curvature κ_g. We can ask whether the 1-form along a curve γ given by

$$\omega_\gamma = \sqrt{|\kappa_g'|}\, ds$$

is a conformal invariant. More precisely, let $\Psi : (M_1, h_1) \to (M_2, h_2)$ be a conformal map and let γ_1 be a curve on M_1; is it true that

$$\Psi^*(\omega_{\Psi(\gamma_1)}) = \omega_{\gamma_1} \ . \tag{4.1}$$