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Kapitel: §6. Normal form and inversive curvature

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Combining this with 4.5 yields

$$\mathbf{n}(\tau(f)) - \tau(\mathbf{n}(f)) = [\mathbf{n}, \tau](f) = \kappa_1\tau(f) + \mu_1\mathbf{n}(f) = \kappa_1\tau(f) + \mathbf{n}\tau(f)$$

which gives 4.6 as required.

§5. A GENERALIZED FOUR VERTEX THEOREM

The curves of constant curvature in the round 2-sphere S^2, the upper half plane H^2 (hyperbolic space), and the Euclidean plane $\mathbf{R}^2$ are just the circles. Moreover, the stereographic projection $p: S^2 \to \mathbf{R}^2$ and the inclusion $i: H^2 \to \mathbf{R}^2$ both preserve these circles. Thus theorem 4.2 says that our form

$$\omega = \sqrt{|\kappa'(s)|}\, ds$$

along a curve γ in $\mathbf{R}^2$ pulls back via p or i to the form

$$\omega = \sqrt{|\kappa_g'(s)|}\, ds$$

along the corresponding curve γ', where here $\kappa_g(s)$ and s refer to the geodesic curvature and arc-length of γ' in the metric for S^2 or H^2. Thus we obtain the four vertex theorem for S^2 and H^2. It follows that the four vertex theorem holds for all complete simply connected Riemannian surfaces of constant curvature. Finally if γ is a null-homotopic smooth simple closed curve on an arbitrary complete Riemannian surface M of constant curvature, then γ lifts one-to-one to a smooth simple closed curve with the same number of vertices on the simply connected universal cover of M. Once again it follows that the number of vertices is at least four.

Remark 5.1. Interestingly, simple closed homotopically non-trivial curves in the real projective plane always have at least three vertices [17]. Note that in non-orientable surfaces the number of honest vertices of a closed curve need not necessarily be even, since here geodesic curvature is only defined up to a sign.

§6. NORMAL FORM AND INVERSIVE CURVATURE

Let p be a non-vertex point of an oriented curve γ. Since the subgroup of Euclidean motions in G acts transitively on the points of $\mathbf{R}^2$ and the unit tangent vectors at these points, we may assume that the point $p \in \gamma$ which interests us lies at the origin with tangent vector in the positive x-direction. Then the curve has the following Taylor series at the origin

$$y = Ax^2 + Bx^3 + Cx^4 + Dx^5 + O(x^6) . \tag{6.1}$$

The coefficients in 6.1 can be expressed in terms of the Euclidean curvature at $p \in \gamma$ and its derivatives with respect to Euclidean arc-length according to the following formulas.

$$A = \frac{1}{2}\,\kappa$$

$$B = \frac{1}{6}\,\kappa'$$

$$C = \frac{1}{24}\,(\kappa'' + 3\kappa^3)$$

$$D = \frac{1}{120}\,(\kappa''' + 19\kappa^2\kappa') .$$

Next we use the non-Euclidean motions of G to further normalize the equation for γ at p. Writing

$$F(z) = -y + Ax^2 + Bx^3 + \dots$$

we have

$$\gamma = \{z \in \mathbf{C} \,|\, F(z) = 0\}, \text{ so that } g^{-1}(\gamma) = \{g^{-1}(z) \,|\, F(z) = 0\}$$
$$= \{z \,|\, F(g(z)) = 0\} .$$

It is rather tedious to calculate the following sequence of transformations of the equations and so we suppress the algebraic work. Since we assume that p is not a vertex we have $B = \kappa'/6 \neq 0$. The substitution

$$z \mapsto \frac{z}{1 - iAz}$$

replaces 6.1 by a new series, which when solved for y yields the Taylor series

$$y = Bx^3 + (C - A^3)x^4 + (D - 4A^2B)x^5 + O(x^6) . \tag{6.2}$$

Next the substitution

$$z \mapsto \frac{z}{1 + \dfrac{C - A^3}{B}\, z}$$

applied to 6.2 yields, after solving for y,

$$y = Bx^3 + \left(D - 4A^2B - \frac{(C-A^3)^2}{B}\right) x^5 + O(x^6) \tag{6.3}$$

Finally the substitution

$$z \mapsto \frac{1}{\sqrt{|6B|}} z$$

applied to 6.3 yields

$$y = \pm \frac{x^3}{6} + Q\frac{x^5}{60} + O(x^6) \tag{6.4}$$

where $\pm$ is the sign of κ', and

$$Q = \frac{5}{3}\left\{\frac{D - 4A^2B}{B^2} - \frac{(C-A^3)^2}{B^3}\right\}$$

Expressing Q in terms of the Euclidean curvature we have

$$Q = \frac{4(\kappa''' - \kappa^2\kappa')\kappa' - 5\kappa''^2}{8\kappa'^3}$$

In particular our calculation shows that there exists an orientation preserving group element g^{-1} moving an arbitrary oriented non-vertex point to the origin in such a way that the image curve has for its Taylor expansion the normal form

$$y = \pm \frac{x^3}{6} + O(x^5)$$

and is oriented in the positive x-direction. The uniqueness of g^{-1} follows by showing the stabilizer of this normal form to be the identity. We omit this calculation. It follows that Q, *the inversive curvature of* γ, is invariant under the group of orientation preserving Möbius transformations.

At a vertex things work out somewhat differently. Assuming that a vertex is *non-degenerate,* in the sense that $\kappa'' \neq 0$ there, one finds, on attempting to imitate the above reduction, that the fourth order term cannot now be eliminated whereas the fifth order term can be removed. One finds that the normal form at a non-degenerate vertex is

$$y = \frac{1}{24}x^4 + \left\{\frac{\kappa''''}{720} - \frac{7}{1440}\kappa^2\kappa'' - \frac{1}{800}\frac{\kappa'''^2}{\kappa''}\right\} \kappa''^{-5/3} x^6 + O(x^7)\,.$$

The stabilizer of the expression $y = x^4/24 + O(x^6)$ has order 2 and is generated by

$$z \mapsto -\bar{z}\,.$$

It follows that the non-degeneracy of a vertex is an invariant of inversive geometry.

§7. The canonical map $g:\gamma \to G$

The considerations of the last section allow us to define a canonical map $g_\gamma : \gamma \to G$ for vertex free curves γ by mapping a point $p \in \gamma$ to $g_\gamma(p) \in G$, which is the unique group element such that $g_\gamma(p)^{-1}$ sends p to the origin and $g_\gamma(p)^{-1}(\gamma)$ has oriented contact of order 4 with the standard curve $y = x^3/6$ at the origin. We note that if $\gamma' = h(\gamma)$ for some $h \in G$, then obviously $g_{\gamma'}(h(p)) = h(g_\gamma(p))$. Of course altering the initial choice of the origin and the axes used there to describe the model will alter g_γ, but only by right multiplication by some fixed element of G. If $\sigma : (\alpha, \beta) \to \mathbf{C}$ is a parametrization of the curve by Euclidean arc-length s, and $\sigma'(s) = e^{i\theta(s)}$, then the curvature of the curve at $\sigma(s)$ is $\theta'(s) = \kappa(s)$, and we have the following explicit formula for g.

$$g(s) = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (\kappa'' - 2i\kappa\kappa')/4\kappa' & 1 \end{pmatrix} \begin{pmatrix} |\kappa'|^{-1/4} & 0 \\ 0 & |\kappa'|^{1/4} \end{pmatrix}$$

The first two factors are Euclidean motions whose inverse puts γ into oriented first order contact with the oriented x-axis. The rest improve the order of contact to 4 as in §6. It is convenient to regard g as a function of the inverse arc-length ν. Now $g(\nu)$ is a curve on the Lie group G, with tangent vector $dg/d\nu$ at $g(\nu)$. Left translation by $g(\nu)^{-1}$ moves this tangent vector to the origin to yield

$$c(\nu) = g(\nu)^{-1} \frac{dg}{d\nu} \tag{7.1}$$

which is a vector in the Lie algebra $sl_2(\mathbf{C})$ of 2 by 2 complex matrices of trace zero. As ν varies $c(\nu)$ inscribes a curve on this Lie algebra. Indeed it is well known (e.g. [13], p. 71) that this curve determines the original curve $g(\nu)$ up to left translation by an arbitrary constant element of G. Here is an explicit formula for the curve $c(\nu)$. It is easy but rather tedious to verify it.