Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 36 (1990)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE INVERSIVE DIFFERENTIAL GEOMETRY OF PLANE CURVES

Autor: Cairns, G. / Sharpe, R. W.

Kapitel: §8. Relation with Cartan's moving frames

DOI: https://doi.org/10.5169/seals-57907

$$c(\nu) = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \quad \text{where} \quad T = \frac{1}{2}\operatorname{sgn}(\kappa')\,(Q-i)$$

and Q is as in § 6. It follows that the inversive curvature Q determines the curve up to an orientation preserving inversive automorphism.

§ 8. Relation with Cartan's moving frames

Let us sketch a more usual way of obtaining a Frenet lift. The connection with the Schwartzian described here can be found, for example, in Cartan's book [4] and very succintly in [7]. The canonical line bundle

$$p : \xi \to \mathbf{P}^1(\mathbf{C})$$

has a pedestrian description (away from the zero-section) as:

$$\begin{array}{ccc} (z_1, z_2) \in \xi - \{\text{zero section}\} = \mathbf{C}^2 - \{0\} \\ \updownarrow \quad {\scriptstyle p} \downarrow \\ z = \dfrac{z_1}{z_2} \leftrightarrow [z_1, z_2] \in \mathbf{P}^1(\mathbf{C}) \end{array}$$

Let $\sigma : (\alpha, \beta) \to \mathbf{R}^2 \subset \mathbf{P}^1(\mathbf{C})$ be a curve; we choose an arbitrary lift $\hat{\sigma} = (z_1(t), z_2(t))$ and set $f_1 = \lambda\hat{\sigma}, f_2 = \dot{f}_1 = \dot{\lambda}(z_1, z_2) + \lambda(\dot{z}_1, \dot{z}_2)$, where $\cdot = \dfrac{d}{dt}$. Thus (f_1, f_2) is a frame in $\mathbf{C}^2$. We try to choose λ so that this frame has area 1. The condition on λ is:

$$\begin{aligned} 1 = \operatorname{Area}(f_1, f_2) &= \operatorname{Area}\big(\lambda(z_1(t), z_2(t)),\ \lambda(\dot{z}_1, \dot{z}_2)\big) \\ &= \lambda^2(z_1\dot{z}_2 - z_2\dot{z}_1), \text{ or } 1 = -(\lambda z_2)^2\dot{z}\ . \end{aligned}$$

Thus $\lambda = \dfrac{i}{z_2\sqrt{\dot{z}}}$ will do, and we have

$$f_1 = \frac{i}{\sqrt{\dot{z}}}\,(z, 1)\ ,$$

$$\text{and} \quad f_2 = \dot{f}_1 = -\frac{1}{2} i \ddot{z}\, \dot{z}^{-3/2}(z, 1) + i\dot{z}^{-1/2}(z, 0)\ .$$

Finally a calculation shows that $\dot{f}_2 = Sf_1$, where $S = \dfrac{3}{4}\ddot{z}^2\dot{z}^{-2} - \dfrac{1}{2}\dddot{z}\,\dot{z}^{-1}$.

Of course S is the *Schwartzian derivative* which this calculation interprets as a "curvature" of σ. Now the Schwartzian S depends on the particular parametrization which is used for the curve. For our purposes we wish to use *inversive arc-length* as the parameter, so that the "curvature" S becomes an *intrinsic* invariant of the curve in inversive geometry. And in this case it turns out that S has constant imaginary part. To see this we describe S in terms of the more familiar Euclidean curvature and its derivatives with respect to Euclidean arc-length.

The Euclidean and inversive arc-lengths are related by the equation:

$$d\nu = |\kappa'|^{1/2} ds, \quad \text{where} \quad ' = \frac{d}{ds}. \text{ Thus:}$$

$$\dot{z} = z'|\kappa'|^{-1/2} = e^{i\theta}|\kappa'|^{-1/2},$$

$$\ddot{z} = \operatorname{sgn}(\kappa')e^{i\theta}\left\{-\frac{1}{2}\frac{\kappa''}{\kappa'^2} + i\frac{\kappa}{\kappa'}\right\},$$

$$\dddot{z} = \operatorname{sgn}(\kappa')e^{i\theta}|\kappa'|^{-1/2}\left\{-\frac{1}{2}\frac{\kappa'''}{\kappa'^2} + \frac{\kappa''^2}{\kappa'^3} - \frac{\kappa^2}{\kappa'} + i\left(1 - \frac{3}{2}\frac{\kappa\kappa''}{\kappa'^2}\right)\right\}$$

Using these expressions we can calculate the Schwartzian as:

$$\frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 - \frac{1}{2}\frac{\dddot{z}}{\dot{z}} = \operatorname{sgn}(\kappa')\left\{\frac{4(\kappa''' - \kappa^2\kappa')\kappa' - 5\kappa''^2}{16\kappa'^3} - \frac{i}{2}\right\} = \frac{1}{2}\operatorname{sgn}(\kappa')\,(Q - i)$$

Regarding the vectors f_1 and f_2 as column vectors, we obtain a 2 by 2 matrix $h = (f_2, f_1) \in G$, and according to the calculation above we have:

$$(\dot{f}_2, \dot{f}_1) = (f_2, f_1)\begin{pmatrix} 0 & 1 \\ S & 0 \end{pmatrix}$$

Thus $h(\nu)$ and $g(\nu)$ (cf. §7) are equal up to left multiplication by a constant element of G. This interprets Cartan's canonical frame (f_1, f_2) as the unique frame (up to a constant element of G) forming the columns of a matrix in G which moves the standard curve $y = x^3/6$ to the given curve with contact up to 4th order at the given point.

§9. Loxodromes

To calculate the curves with Q constant we solve the equation:

$$\frac{dg}{d\nu} = g\begin{pmatrix} 0 & 1 \\ \frac{1}{2}\varepsilon(Q - i) & 0 \end{pmatrix}, \quad \text{where } \varepsilon = \operatorname{sgn}(\kappa').$$