

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 36 (1990)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE INVERSIVE DIFFERENTIAL GEOMETRY OF PLANE CURVES
Autor: Cairns, G. / Sharpe, R. W.
Kapitel: §8. Relation with Cartan's moving frames
DOI: <https://doi.org/10.5169/seals-57907>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 09.03.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$$c(v) = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \quad \text{where } T = \frac{1}{2} \operatorname{sgn}(\kappa') (Q - i)$$

and Q is as in §6. It follows that the inversive curvature Q determines the curve up to an orientation preserving inversive automorphism.

§8. RELATION WITH CARTAN'S MOVING FRAMES

Let us sketch a more usual way of obtaining a Frenet lift. The connection with the Schwartzian described here can be found, for example, in Cartan's book [4] and very succinctly in [7]. The canonical line bundle

$$p: \xi \rightarrow \mathbf{P}^1(\mathbf{C})$$

has a pedestrian description (away from the zero-section) as:

$$\begin{array}{ccc} (z_1, z_2) \in \xi - \{\text{zero section}\} = \mathbf{C}^2 - \{0\} & & \\ \downarrow & p \downarrow & \\ z = \frac{z_1}{z_2} \leftrightarrow [z_1, z_2] \in \mathbf{P}^1(\mathbf{C}) & & \end{array}$$

Let $\sigma: (\alpha, \beta) \rightarrow \mathbf{R}^2 \subset \mathbf{P}^1(\mathbf{C})$ be a curve; we choose an arbitrary lift $\hat{\sigma} = (z_1(t), z_2(t))$ and set $f_1 = \lambda \hat{\sigma}$, $f_2 = \dot{f}_1 = \lambda(z_1, z_2) + \lambda(\dot{z}_1, \dot{z}_2)$, where $\dot{} = \frac{d}{dt}$. Thus (f_1, f_2) is a frame in \mathbf{C}^2 . We try to choose λ so that this frame has area 1. The condition on λ is:

$$\begin{aligned} 1 &= \text{Area}(f_1, f_2) = \text{Area}(\lambda(z_1(t), z_2(t)), \lambda(\dot{z}_1, \dot{z}_2)) \\ &= \lambda^2(z_1 \dot{z}_2 - z_2 \dot{z}_1), \quad \text{or } 1 = -(\lambda z_2)^2 \dot{z} \end{aligned}$$

Thus $\lambda = \frac{i}{z_2 \sqrt{\dot{z}}}$ will do, and we have

$$\begin{aligned} f_1 &= \frac{i}{\sqrt{\dot{z}}} (z, 1), \\ \text{and } f_2 &= \dot{f}_1 = -\frac{1}{2} i \ddot{z} \dot{z}^{-3/2} (z, 1) + i \dot{z}^{-1/2} (z, 0). \end{aligned}$$

Finally a calculation shows that $\dot{f}_2 = S f_1$, where $S = \frac{3}{4} \ddot{z}^2 \dot{z}^{-2} - \frac{1}{2} \ddot{z} \dot{z}^{-1}$.

Of course S is the *Schwartzian derivative* which this calculation interprets as a "curvature" of σ . Now the Schwartzian S depends on the particular parametrization which is used for the curve. For our purposes we wish to use

inversive arc-length as the parameter, so that the “curvature” S becomes an *intrinsic* invariant of the curve in inversive geometry. And in this case it turns out that S has constant imaginary part. To see this we describe S in terms of the more familiar Euclidean curvature and its derivatives with respect to Euclidean arc-length.

The Euclidean and inversive arc-lengths are related by the equation:

$$dv = |\kappa'|^{1/2} ds, \quad \text{where } ' = \frac{d}{ds}. \text{ Thus:}$$

$$\dot{z} = z' |\kappa'|^{-1/2} = e^{i\theta} |\kappa'|^{-1/2},$$

$$\ddot{z} = \text{sgn}(\kappa') e^{i\theta} \left\{ -\frac{1}{2} \frac{\kappa''}{\kappa'^2} + i \frac{\kappa}{\kappa'} \right\},$$

$$\ddot{\dot{z}} = \text{sgn}(\kappa') e^{i\theta} |\kappa'|^{-1/2} \left\{ -\frac{1}{2} \frac{\kappa'''}{\kappa'^2} + \frac{\kappa''^2}{\kappa'^3} - \frac{\kappa^2}{\kappa'} + i \left(1 - \frac{3}{2} \frac{\kappa \kappa''}{\kappa'^2} \right) \right\}$$

Using these expressions we can calculate the Schwartzian as:

$$\frac{3}{4} \left(\frac{\ddot{\dot{z}}}{\dot{z}} \right)^2 - \frac{1}{2} \frac{\ddot{\dot{z}}}{\dot{z}} = \text{sgn}(\kappa') \left\{ \frac{4(\kappa''' - \kappa^2 \kappa') \kappa' - 5\kappa''^2}{16\kappa'^3} - \frac{i}{2} \right\} = \frac{1}{2} \text{sgn}(\kappa') (Q - i)$$

Regarding the vectors f_1 and f_2 as column vectors, we obtain a 2 by 2 matrix $h = (f_2, f_1) \in G$, and according to the calculation above we have:

$$(f_2, f_1) = (f_2, f_1) \begin{pmatrix} 0 & 1 \\ S & 0 \end{pmatrix}$$

Thus $h(v)$ and $g(v)$ (cf. §7) are equal up to left multiplication by a constant element of G . This interprets Cartan’s canonical frame (f_1, f_2) as the unique frame (up to a constant element of G) forming the columns of a matrix in G which moves the standard curve $y = x^3/6$ to the given curve with contact up to 4th order at the given point.

§9. LOXODROMES

To calculate the curves with Q constant we solve the equation:

$$\frac{dg}{dv} = g \begin{pmatrix} 0 & 1 \\ \frac{1}{2} \varepsilon(Q - i) & 0 \end{pmatrix}, \quad \text{where } \varepsilon = \text{sgn}(\kappa')$$