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inversive arc-length as the parameter, so that the "curvature" S becomes an *intrinsic* invariant of the curve in inversive geometry. And in this case it turns out that S has constant imaginary part. To see this we describe S in terms of the more familiar Euclidean curvature and its derivatives with respect to Euclidean arc-length.

The Euclidean and inversive arc-lengths are related by the equation:

$$d\nu = |\kappa'|^{1/2} ds, \quad \text{where} \quad ' = \frac{d}{ds}. \text{ Thus:}$$

$$\dot{z} = z'|\kappa'|^{-1/2} = e^{i\theta}|\kappa'|^{-1/2},$$

$$\ddot{z} = \operatorname{sgn}(\kappa')e^{i\theta}\left\{-\frac{1}{2}\frac{\kappa''}{\kappa'^2} + i\frac{\kappa}{\kappa'}\right\},$$

$$\dddot{z} = \operatorname{sgn}(\kappa')e^{i\theta}|\kappa'|^{-1/2}\left\{-\frac{1}{2}\frac{\kappa'''}{\kappa'^2} + \frac{\kappa''^2}{\kappa'^3} - \frac{\kappa^2}{\kappa'} + i\left(1 - \frac{3}{2}\frac{\kappa\kappa''}{\kappa'^2}\right)\right\}$$

Using these expressions we can calculate the Schwartzian as:

$$\frac{3}{4}\left(\frac{\ddot{z}}{\dot{z}}\right)^2 - \frac{1}{2}\frac{\dddot{z}}{\dot{z}} = \operatorname{sgn}(\kappa')\left\{\frac{4(\kappa''' - \kappa^2\kappa')\kappa' - 5\kappa''^2}{16\kappa'^3} - \frac{i}{2}\right\} = \frac{1}{2}\operatorname{sgn}(\kappa')\,(Q - i)$$

Regarding the vectors f_1 and f_2 as column vectors, we obtain a 2 by 2 matrix $h = (f_2, f_1) \in G$, and according to the calculation above we have:

$$(\dot{f}_2, \dot{f}_1) = (f_2, f_1)\begin{pmatrix} 0 & 1 \\ S & 0 \end{pmatrix}$$

Thus $h(\nu)$ and $g(\nu)$ (cf. §7) are equal up to left multiplication by a constant element of G. This interprets Cartan's canonical frame (f_1, f_2) as the unique frame (up to a constant element of G) forming the columns of a matrix in G which moves the standard curve $y = x^3/6$ to the given curve with contact up to 4th order at the given point.

§9. Loxodromes

To calculate the curves with Q constant we solve the equation:

$$\frac{dg}{d\nu} = g\begin{pmatrix} 0 & 1 \\ \frac{1}{2}\varepsilon(Q - i) & 0 \end{pmatrix}, \quad \text{where} \quad \varepsilon = \operatorname{sgn}(\kappa')$$

and use the fact that $\sigma(v) = g(v)\cdot 0$ to obtain the curve (where $\cdot$ refers to the action of $Sl_2(\mathbf{C})$ on $\mathbf{C}$ by Möbius transformations).

Now

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{2}\varepsilon(Q-i) & 0 \end{pmatrix} = A\begin{pmatrix} \xi & 0 \\ 0 & -\xi \end{pmatrix} A^{-1},$$

where

$$\xi = \pm\frac{1}{\sqrt{2}}(1+Q^2)^{1/4}\sqrt{\varepsilon}\, e^{-\frac{1}{2}i\tan^{-1}\left(\frac{1}{Q}\right)}$$

and

$$A = \frac{1}{\sqrt{-2\xi}}\begin{pmatrix} 1 & 1 \\ \xi & -\xi \end{pmatrix}.$$

Thus $\frac{dgA}{dv} = gA\begin{pmatrix} \xi & 0 \\ 0 & -\xi \end{pmatrix}$ and hence $gA = C\begin{pmatrix} e^{\xi v} & 0 \\ 0 & e^{-\xi v} \end{pmatrix}$, where C is an invertible constant 2 by 2 matrix. Since $A\cdot -1 = 0$ we have

$$g\cdot 0 = -C\cdot(e^{2\xi v})$$

which is a linear fractional image of an equiangular spiral.

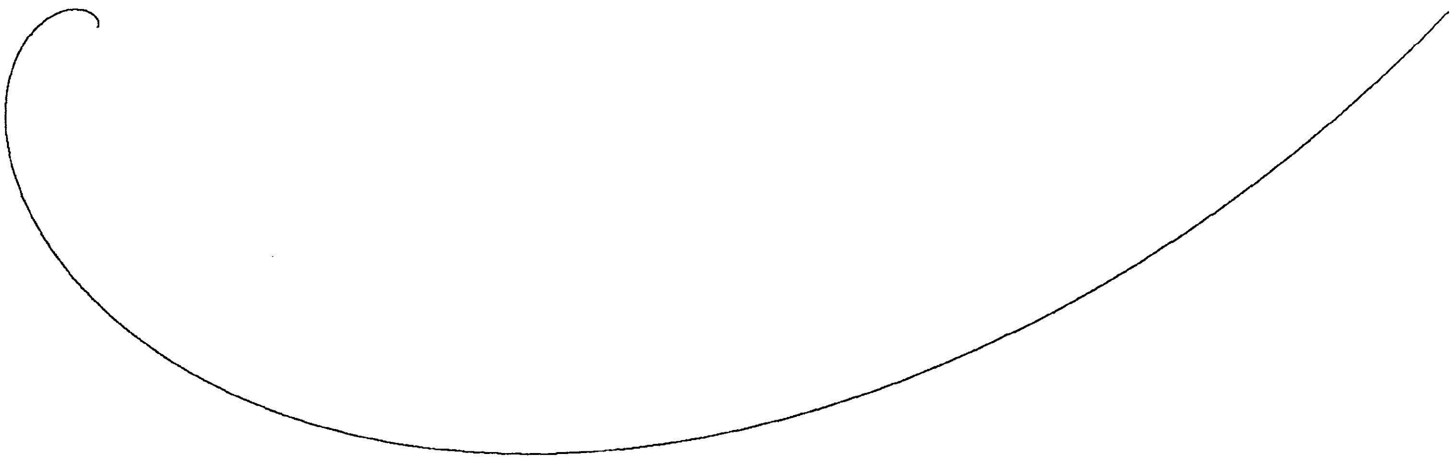

FIGURE 9.1

In particular, curves for which $Q = 0$, which we may call *inversive geodesics,* have

$$2\xi = \pm\sqrt{2\varepsilon}\, e^{-\frac{1}{2}\frac{\pi}{2}i} = \pm 1 \pm i$$

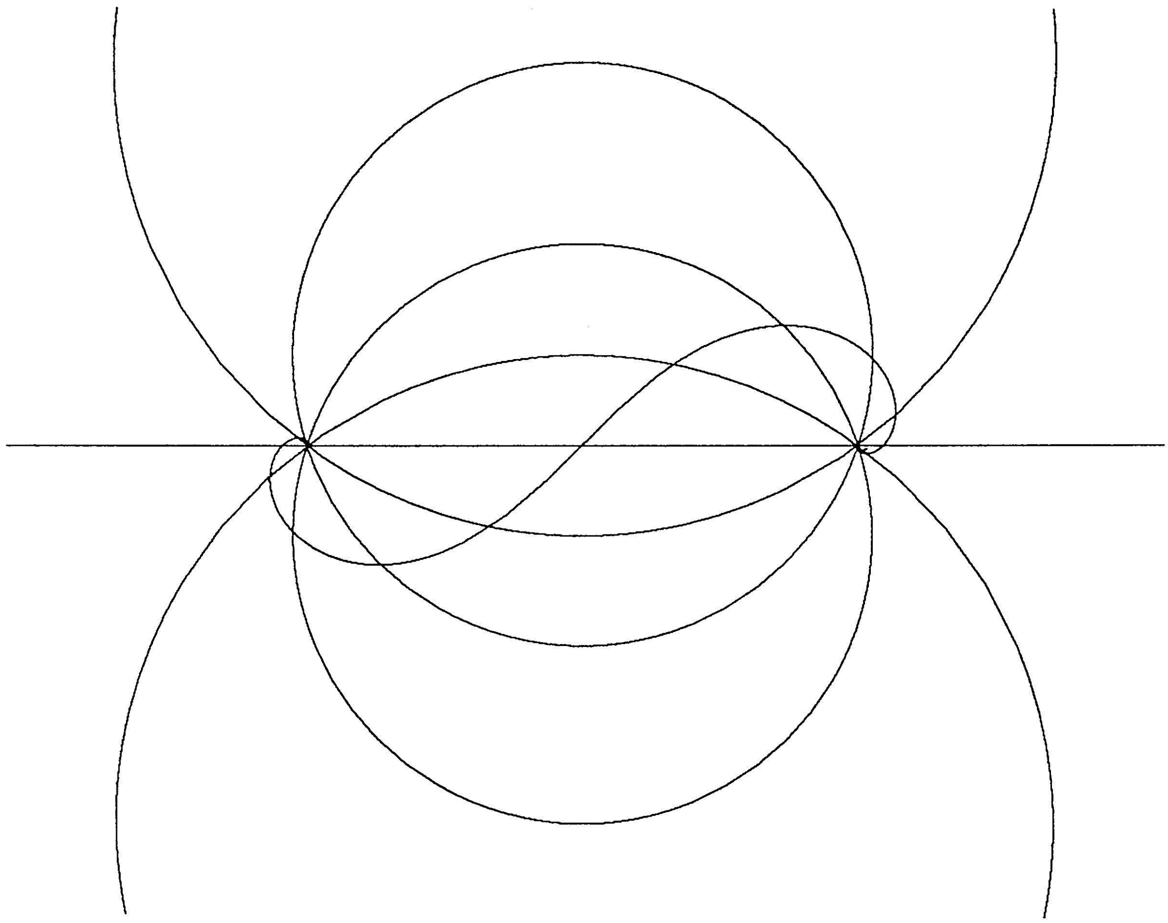

FIGURE 9.2

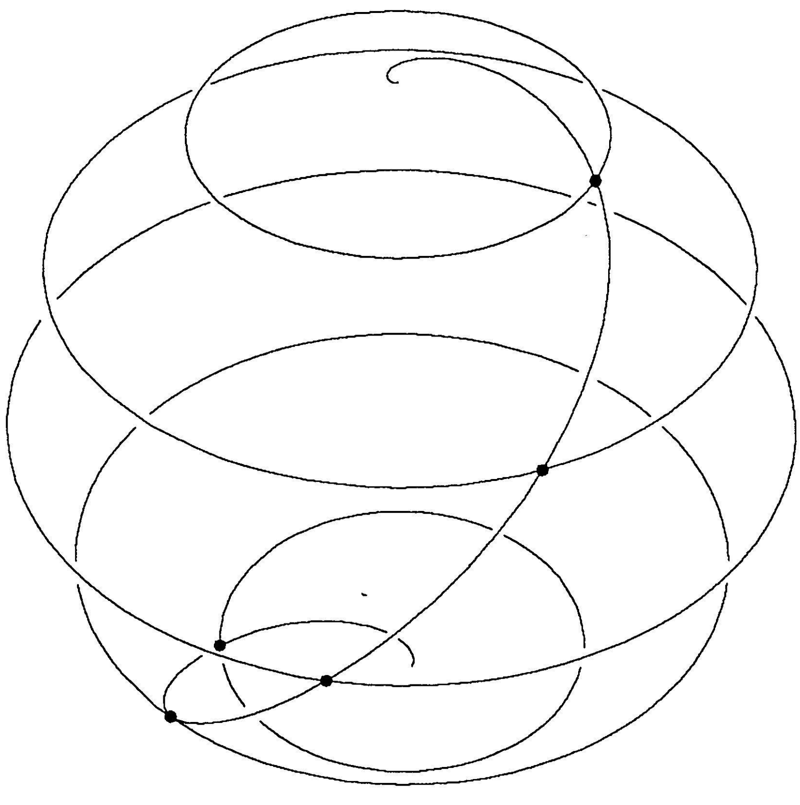

FIGURE 9.3

and are linear fractional images of the equiangular spiral with angle $\pm\pi/4$ given by

$$\sigma(v) = e^{(1 \pm i)v} = e^{v} e^{\pm iv} .$$

We note in particular that the inverse length of "one loop" of the inversive geodesic is 2π. Figure 9.1 is a picture of one such loop.

Equiangular spirals have two accumulation points, the poles, one at the origin and the other at infinity. These poles determine the family of circles through them (straight lines in this case) as well as a second family of circles orthogonal to the first. The equiangular spiral meets each family in fixed angles. The same is true for linear fractional images of this configuration, and with the same angles.

The connection between Q and the angle $\varphi \in (0, \pi/2)$ between the loxodrome and its first family of circles is given by

$$\tan\varphi = Q + \sqrt{Q^2 + 1} .$$

In figure 9.2, we show the inversive geodesic with poles at ± 1 together with its first family of circles.

In figure 9.3 we see the loxodrome again, in a perspective view this time, thrown up onto the two-sphere by the inverse of stereographic projection, along with its second family of circles.

We remark that it seems to be impossible to show the inverse geodesic in such a way as to allow more than one or two loops to appear to the eye, while at the same time allowing no distortion of the figure. This may account for a number of distorted diagrams of this loxodrome which have appeared in the literature. Of course one can picture many loops of some equiangular spirals, say with $Q \gg 0$. At the other extreme with $Q \ll 0$ we have a circumstance for which, in any scale, the corresponding equiangular spiral appears to the eye to be a straight line issuing from the origin. However as one "zooms" in or out this "straight line" appears to rotate about the origin.

§ 10. THE COMPLEX OF GEOMETRIC FORMS ON A CURVE IN $\mathbf{R}^2$

Among the various forms on a curve in $\mathbf{R}^2$, some, such as ω and Q, can be thought of as arising from the local way in which the curve is embedded in $\mathbf{R}^2$; that is they arise from the local geometric nature of the embedding and are invariant under Möbius transformations. These are the "smooth local