Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: DESCARTES' THEOREM IN n DIMENSIONS
Autor: Grünbaum, Branko / Shephard, G. C.
DOI: https://doi.org/10.5169/seals-58726

L'Enseignement Mathématique, t. 37 (1991), p. 11-15

DESCARTES' THEOREM IN n DIMENSIONS

by Branko GRÜNBAUM[1]) and G. C. SHEPHARD

The elementary and beautiful theorem known as Descartes' Theorem was discovered in the seventeenth century and is stated in Descartes' *De Solidorum Elementis.* The manuscript was lost, however, and we only know of its contents because a copy made by Leibnitz was discovered in the Royal Library of Hanover in 1860. A transcription and translation of this manuscript, together with comments, can be found in Federico's fascinating account of the work [2].

Descartes proved his theorem for convex polyhedra by observing that the sum of the "exterior" angles is equal to "eight solid right angles". Simpler proofs, depending essentially upon Euler's theorem for polyhedra are now known, and Descartes' Theorem has been extended to (possibly non-convex) *elementary polyhedra,* see [5]. It has also been dualised, see [4].

In [5, p. 343], Hilton and Pedersen say "...there will be no straightforward generalization to higher dimensions of Descartes' formula for the total angular defect of a polyhedron... since this defect ceases in higher dimensions to be a topological invariant". If only "defects" (or *deficiencies* as we prefer to call them) at the vertices are considered, such a statement is undoubtedly true; but there is a straightforward generalisation if deficiencies at faces of other dimensions are taken into account. This was stated in [8] for convex polytopes. Here we generalise Descartes' Theorem to elementary polyhedra (defined below) of arbitrary dimension and Euler characteristic, besides giving a new and much simpler proof.

Before stating this result we must be clear about the kind of polyhedra or polytopes to which it applies. An $(n-1)$-dimensional *convex polytope* is any bounded set of E^{n-1} which has non-empty $(n-1)$-dimensional interior and can be expressed as the intersection of a finite number of closed halfspaces. A family $\{F_1, F_2, \ldots, F_r\}$ of $(n-1)$-dimensional convex polytopes situated in E^n is said to form an *elementary polytope* P (of n dimensions) if

[1]) Research supported in part by NSF grant DMS-8620181.

(i) for each i and j, the intersection $F_i \cap F_j$ is either empty or is a face of each of F_i and F_j; and

(ii) the union of all the F_i is an $(n-1)$-dimensional manifold.

The sets F_i are called the $(n-1)$-faces (or facets) of P, and the other faces of P are the faces of the F_i. Two features of elementary polytopes must be stressed: faces have no mutual intersections other than at their boundaries, and the manifold may have arbitrarily high Euler characteristic. Elementary polytopes have been considered in the literature under a variety of other names; for example, in the 2-dimensional case they are called "polyhedral maps" in [1], while in the terminology of [3] they are cell-complexes which are $(n-1)$-dimensional manifolds.

Throughout we shall use *absolute angle measure.* Here an angle, in any number of dimensions, is measured as a fraction of the "total" angle at a point, that is, the angle subtended at a point by a sphere centred at that point. Thus a right-angle is $\frac{1}{4}$, and the (solid) angle at a vertex of a cube is $\frac{1}{8}$. With this method of measuring angles, many formulae are greatly simplified. In particular, let P be a convex three-dimensional polytope (polyhedron) and $s(V_i)$ be the sum of the (plane) angles at a vertex V_i of P of all the 2-faces of P incident with V_i. Then $\delta(V_i) = 1 - s(V_i)$ is called the (angle) deficiency of P at V_i, and the classical form of Descartes' Theorem states

$$\sum_i \delta(V_i) = 2 .$$

For elementary polyhedra P of higher genus, the same equality holds with the right side replaced by $\chi(P)$, the Euler characteristic of P (see, for example, [5], [6]).

We now state and prove the analogous theorem in four dimensions. At each vertex V_i of a four-dimensional elementary polytope P, if we denote by $s(V_i)$ the sum of the (solid) angles at V_i of all the 3-faces of P incident with V_i, then

$$\delta(V_i) = 1 - s(V_i)$$

is called the (angle) *deficiency* of P at vertex V_i. In a similar way, if we denote by $s(E_j)$ the sum of the (dihedral) angles at an edge E_j of all the 3-faces of P incident with E_j, then

$$\delta(E_j) = 1 - s(E_j)$$

is called the *deficiency* of P at the edge E_j.

Descartes' Theorem in Four Dimensions. *Let P be an elementary polytope in E^4. Then*

(1) $$\sum_i \delta(V_i) - \sum_j \delta(E_j) = \chi(P)$$

where the first summation is over all the vertices V_i of P, the second summation is over all the edges E_j of P, and $\chi(P)$ is the Euler characteristic of P.

To illustrate this theorem, consider the boundary B of the 4-dimensional (hyper)cube. This is a 3-manifold with $\chi(B) = 0$; it consists of 8 cubes of dimension 3, meeting by fours at each of the 16 vertices V_i of B, and by threes at each of the 32 edges E_j of B. By the above,

$$\delta(V_i) = 1 - 4 \times \frac{1}{8} = \frac{1}{2} \quad \text{and} \quad \delta(E_j) = 1 - 3 \times \frac{1}{4} = \frac{1}{4},$$

and (1) becomes

$$16 \times \frac{1}{2} - 32 \times \frac{1}{4} = 0 = \chi(P) .$$

Proof of the theorem. We make use of the three-dimensional form of Gram's Theorem (see [3], Section 14.1). Denoting by $s_0(F_k)$ the sum of the solid angles at the vertices of the 3-face F_k, by $s_1(F_k)$ the sum of the dihedral angles at the edges of F_k, and by $s_2(F_k)$ the sum of the "solid" angles at the 2-faces of F_k (which is equal, by convention, to $\frac{1}{2} f_2(F_k)$, where $f_2(F_k)$ is the number of 2-faces of F_k), then Gram's Theorem is the statement

(2) $$s_0(F_k) - s_1(F_k) + s_2(F_k) = 1 .$$

Various elementary proofs of this result are known, see for example [7].

For $i = 0, 1, 2, 3$, write $f_i(P)$ for the number of i-faces of P, and for $i = 0, 1, 2$, write $s_i(P) = \sum_k s_i(F_k)$, where summation is over the $f_3(P)$ 3-faces of P. In particular, $s_2(P) = f_2(P)$ since each 3-face F_k contributes $\frac{1}{2} f_2(P)$ to the sum, and each 2-face belongs to exactly two 3-faces. Summing (2) over all the 3-faces of P we obtain

(3) $$s_0(P) - s_1(P) + f_2(P) = f_3(P) .$$

However, by definition

$$\sum_i \delta(V_i) = f_0(P) - s_0(P)$$
$$\sum_j \delta(E_j) = f_1(P) - s_1(P) ,$$

so (3) becomes

$$(f_0(P) - \sum_i \delta(V_i)) - (f_1(P) - \sum_j \delta(E_j)) + f_2(P) = f_3(P)$$

or,

$$\sum_i \delta(V_i) - \sum_j \delta(E_j) = f_0(P) - f_1(P) + f_2(P) - f_3(P) = \chi(P)$$

which is (1), and the theorem is proved.

An exactly analogous argument holds in n dimensions (for all $n \geqslant 3$). In the proof we use the $(n-1)$-dimensional form of Gram's Theorem (see [3], [7]) for each $(n-1)$-face F_k of P:

$$s_0(F_k) - s_1(F_k) + s_2(F_k) - \dots + (-1)^{n-2} s_{n-2}(F_k) = (-1)^n$$

with a notation analogous to that used in (2). This leads to the statement:

DESCARTES' THEOREM IN n DIMENSIONS. *Let P be an elementary polytope in E^n, let $\delta(F_i^m) = 1 - s(F_i^m)$ be the deficiency of P at the m-face F_i^m of P $(m = 0, 1, \dots, n-3)$, and let $\delta_m(P) = \sum_i \delta(F_i^m)$, where summation is over all the m-faces F_i^m of P. Then*

$$\sum_{m=0}^{n-3} (-1)^m \delta_m(P) = \chi(P) ,$$

where $\chi(P)$ is the Euler characteristic of P.

REFERENCES

[1] BARNETTE, D. W., P. GRITZMANN and R. HÖHNE. On valences of polyhedra. (To appear.)

[2] FEDERICO, P. J. Descartes on polyhedra, a study of the *De Solidorum Elementis* (Volume 4 in *Sources in the History of Mathematics and Physical Sciences*), Springer-Verlag, New York-Heidelberg-Berlin 1982.

[3] GRÜNBAUM, B. *Convex Polytopes.* Wiley, 1967.

[4] GRÜNBAUM, B. and G. C. SHEPHARD. A dual for Descartes' theorem on polyhedra. *Math. Gazette 71* (1987), 214-216.

[5] HILTON, P. and J. PEDERSEN. Descartes, Euler, Poincaré, Pólya – and polyhedra. *L'Enseignement Math. 27* (1981), 327-343.

[6] Hilton, P. and J. Pedersen. Duality and Descartes' deficiency. *Computers Math. Applic. 17* (1989), 73-88.

[7] Shephard, G. C. An elementary proof of Gram's theorem for convex polytopes. *Canad. J. Math. 19* (1967), 1214-1217.

[8] —— Angle deficiencies for convex polytopes. *J. London Math. Soc. 43* (1968), 325-336.

(Reçu le 5 juin 1990)

Branko Grünbaum

University of Washington GN-50
Seattle, WA 98195, USA

G. C. Shephard

University of East Anglia
Norwich NR4 7TJ, England

vide-leer-empty