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## THE EVALUATION OF SELBERG CHARACTER SUMS

by Ronald J. EVANS

ABSTRACT. The evaluations of Selberg character sums conjectured on p. 207 of *Enseignement Math.* 27 (1981) are proved.

### §1. INTRODUCTION

Many of the classical special functions over  $\mathbf{C}$  have character sum analogs over finite fields. For example, the Gauss and Jacobi sums defined in (1.1) are analogs of the gamma and beta integrals

$$\Gamma(a) = \int_0^\infty e^{-x} x^a \frac{dx}{x}, \quad \beta(a, b) = \int_0^1 x^a (1-x)^b \frac{dx}{x(1-x)}.$$

Some identities for character sums over finite fields seem more difficult to prove than their classical counterparts; compare, e.g., the Hasse-Davenport product formula for Gauss sums [7, (7)] with the Gauss multiplication formula for gamma functions. The identities for  $n$ -dimensional Selberg character sums given in Theorems 1.1, 1.1a provide further examples. Their counterparts are the well known  $n$ -dimensional Selberg integral extensions of the gamma and beta integral formulas.

The case  $n = 3$  of the Selberg character sum identity in Theorem 1.1 has been used to evaluate a sum connected with the root system  $G_2$  [8]. The case  $n = 2$  is equivalent to an analog of Dixon's summation formula [11, (2.1.5)] involving hypergeometric  ${}_3F_2$  character sums over finite fields. We remark that hypergeometric character sums have been used, e.g., in the computation of the number of points on hypersurfaces [13], [12], in proving congruences for Apéry numbers [14], and in graph theory [6], [9].

Let  $GF(q)$  be a finite field of  $q$  elements, where  $q$  is a power of an odd prime. Fix a multiplicative character  $\tau: GF(q)^* \rightarrow \mathbf{C}^*$  of order  $q - 1$  and a nontrivial additive character  $\psi: GF(q) \rightarrow \mathbf{C}^*$ . Extend  $\tau$  by defining  $\tau(0) = 0$ . Let  $\phi = \tau^{(q-1)/2}$  be the quadratic character on  $GF(q)$ . For all integers  $a, b$ , define the Gauss sums  $G(a)$  and Jacobi sums  $J(a, b)$  by

$$(1.1) \quad G(a) = \sum_{\xi \in GF(q)^*} \tau(\xi)^a \psi(\xi), \quad J(a, b) = \sum_{1 \neq \xi \in GF(q)^*} \tau(\xi)^a \tau(1 - \xi)^b.$$

For integers  $n \geq 0$  and  $a, b, c > 0$ , define the Selberg character sums

$$(1.2) \quad S_n(a, b, c) = \sum_E \tau((-1)^{an} E(0)^a E(1)^b \Delta_E^c) \phi(\Delta_E),$$

$$(1.2a) \quad S_n(a, c) = \sum_E \psi(e_{n-1}) \tau(E(0)^a \Delta_E^c) \phi(\Delta_E),$$

$$(1.2b) \quad S_n(c) = \sum_E \psi(e_{n-1}^2/2 - e_{n-2}) \tau(\Delta_E)^c \phi(\Delta_E),$$

where each sum is over all monic polynomials

$$(1.3) \quad E = E(x) = x^n + e_{n-1}x^{n-1} + e_{n-2}x^{n-2} + \cdots + e_0$$

of degree  $n$  over  $GF(q)$ , and where  $\Delta_E$  denotes the discriminant of  $E$  (with the convention that  $\Delta_E = 1$  when  $\deg(E) \leq 1$ ). Define the following products:

$$(1.4) \quad P_n(a, b, c) = \prod_{j=0}^{n-1} \frac{G(a+jc)G(b+jc)G(c+jc)\bar{G}(a+b+(n-1+j)c)}{qG(c)},$$

$$(1.4a) \quad P_n(a, c) = \prod_{j=0}^{n-1} \frac{G(a+jc)G(c+jc)}{G(c)},$$

$$(1.4b) \quad P_n(c) = \prod_{j=0}^{n-1} \frac{G(c+jc)\phi(2)G((q-1)/2)}{G(c)},$$

where  $\bar{G}$  denotes the complex conjugate of  $G$ .

The object of this paper is to prove Theorems 1.1, 1.1a, and 1.1b below. These results, analogs of  $n$ -dimensional integral formulas of Selberg [3, (1.1), (1.3), (1.2)], [2], verify conjectures made in 1981 [7, (29), (29a), (29b)]. The decisive breakthrough came in 1990 when Anderson [1] proved a somewhat weakened form of Theorem 1.1. The proofs here are based on modifications of the method in [1]. The modifications are designed to handle complications arising from "imprimitive"  $L$ -functions (see §2).

**THEOREM 1.1.** *For all integers  $n, a, b, c > 0$ , if none of*

$$a + b + (n-1+j)c \quad (0 \leq j \leq n-1)$$

*are divisible by  $q-1$ , then  $S_n(a, b, c) = P_n(a, b, c)$ .*

THEOREM 1.1a. For all integers  $n, a, c > 0$ ,  $S_n(a, c) = P_n(a, c)$ .

THEOREM 1.1b. For all integers  $n, c > 0$ ,  $S_n(c) = P_n(c)$ .

Given a monic polynomial  $E$  over  $GF(q)$ , define  $\sigma(E) = 0$  if  $E$  is not squarefree,  $\sigma(E) = 1$  if  $E = 1$ , and otherwise let  $\sigma(E)$  denote the sign of the permutation of the zeros of  $E$  effected by the  $q^{th}$  power automorphism of  $\overline{GF(q)}$ . For odd  $q$ ,  $\sigma(E) = \phi(\Delta_E)$ . If  $\phi(\Delta_E)$  is replaced by  $\sigma(E)$  in the definitions (1.2), (1.2a) of  $S_n(a, b, c)$ ,  $S_n(a, c)$ , then Theorems 1.1 and 1.1a remain valid without the stipulation “ $q$  odd”; the proofs for even  $q$  are virtually the same. This observation is due to Serre; see [1].

The following result is equivalent to Theorem 1.1, as was shown in [10, p. 116].

THEOREM 1.2. For integers  $n, a, b, c > 0$ , if none of  $a + jc$  ( $0 \leq j \leq n - 1$ ) are divisible by  $q - 1$ , or if none of  $b + jc$  ( $0 \leq j \leq n - 1$ ) are divisible by  $q - 1$ , or if none of  $a + b + (n - 1 + j)c$  ( $0 \leq j \leq n - 1$ ) are divisible by  $q - 1$ , then  $S_n(a, b, c) = P_n(a, b, c)$ .

Theorems 1.3 and 1.4 below, analogs of more recent Selberg integral formulas (see [4]), were stated as conjectures in [5]. They are consequences of Theorems 1.1a and 1.1b, respectively, as is shown in [5, Theorems 2.2 and 2.5].

THEOREM 1.3. For all integers  $n, a, b, c > 0$ ,

$$\sum_E \tau(E(0)^a (1 + e_{n-1})^b \Delta_E^c) \phi(\Delta_E) = \begin{cases} \frac{G(-b - na - n(n-1)c)}{G(-b)} P_n(a, c), & \text{if } b \not\equiv 0 \pmod{q-1} \\ \frac{\tau(-1)^{an} G(b)}{G(b + na + n(n-1)c)} P_n(a, c), & \text{if } b + na + n(n-1)c \not\equiv 0 \pmod{q-1}, \end{cases}$$

where the sum is over all polynomials  $E$  of degree  $n$  given by (1.3).

THEOREM 1.4. For  $w \in GF(q)^*$  and all integers  $n, b, c > 0$  with  $b \not\equiv 0 \pmod{q-1}$ ,

$$\sum_E \tau((w + e_{n-1}^2/2 - e_{n-2})^b \Delta_E^c) \phi(\Delta_E) = \tau(w)^{b+n(q-1)/2+cn(n-1)/2} \frac{G(-b - cn(n-1)/2 - n(q-1)/2)}{G(-b)} P_n(c),$$

where the sum is over all polynomials  $E$  of degree  $n$  given by (1.3).

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## §2. $L$ -FUNCTIONS

Throughout this section,  $V$  denotes a *monic* polynomial over  $GF(q)$ , and  $v$  ranges over the distinct monic irreducible factors of  $V$  over  $GF(q)$ . Write

$$(2.1) \quad V = \prod_{v|V} v^{\text{ord}_v V}, \quad F = F_V = \prod_{v|V} v.$$

If no exponent  $\text{ord}_v V$  in (2.1) is divisible by  $q - 1$ , then  $V$  is said to be *primitive*. Note that  $V = 1$  is primitive. For any monic polynomial

$$(2.2) \quad W = W(x) = x^n + w_{n-1}x^{n-1} + w_{n-2}x^{n-2} + \cdots + w_0$$

over  $GF(q)$ , set

$$(2.3) \quad \alpha(W) = w_{n-1}, \quad \beta(W) = w_{n-1}^2/2 - w_{n-2}.$$

Define the  $L$ -functions

$$(2.4) \quad L(t, V) = \sum_W \tau(R(V, W)) t^{\deg W},$$

$$(2.4a) \quad L_1(t, V) = \sum_W \psi(\alpha(W)) \tau(R(V, W)) t^{\deg W},$$

$$(2.4b) \quad L_2(t, V) = \sum_W \psi(\beta(W)) \tau(R(V, W)) t^{\deg W},$$

where in each sum,  $W$  ranges over all monic polynomials over  $GF(q)$ , and  $R(V, W)$  is the resultant of  $V$  and  $W$ . It is easily checked that

$$(2.5) \quad \begin{aligned} L(t, 1) &= (1 - qt)^{-1}, & L_1(t, 1) &= 1, \\ L_2(t, 1) &= 1 + \phi(2)G((q-1)/2)t. \end{aligned}$$

Since the summands in (2.4), (2.4a), (2.4b) are multiplicative in  $W$ , each of the  $L$ -functions has an Euler product expansion. Thus we have the following result.

LEMMA 2.1. Write  $V = GH$  where  $G$  and  $H$  are monic, relatively prime polynomials over  $GF(q)$  with  $G$  primitive and  $H$  a  $(q-1)$ th power. Then

$$(2.6) \quad L(t, V) = L(t, G) \prod_{v|H} (1 - \tau(R(G, v)) t^{\deg v}),$$

$$(2.6a) \quad L_1(t, V) = L_1(t, G) \prod_{v|H} (1 - \psi(\alpha(v)) \tau(R(G, v)) t^{\deg v}),$$