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# ON THE DIFFEOMORPHISM GROUPS OF CERTAIN ALGEBRAIC SURFACES

by Wolfgang EBELING and Christian OKONEK

## INTRODUCTION

It has become clear in the last years that there is an essential difference between the topology and the differential topology of 4-manifolds. The new understanding is based on fundamental ideas and results due to M. Freedman and S.K. Donaldson. The difference between the two categories becomes particularly apparent for the underlying 4-manifolds of complex algebraic surfaces. There are e.g. algebraic surfaces whose underlying topological manifolds carry infinitely many differentiable structures.

The object of this paper is to study in an analogous way the automorphisms of the two categories, i.e. the symmetry groups of 4-manifolds. For the topological category this is well understood. By the result of Freedman [F], simply-connected closed oriented topological 4-manifolds are classified up to homeomorphism by the isomorphism class of a lattice and the Kirby-Siebenmann invariant in $\mathbf{Z}/2$. If $X$ is such a manifold, then the lattice in question is $L = (H_2(X, \mathbf{Z}), q_X)$, where $q_X$ is the quadratic intersection form of $X$. An orientation preserving homeomorphism of $X$ induces an isometry of $L$, and isotopic homeomorphisms induce the same isometry. Therefore there is a natural homomorphism $\psi_{\text{top}}$ from the component group $\text{Top}_+(X)$ of the group of orientation preserving homeomorphisms of $X$ to the orthogonal group $O(L)$ of the lattice $L$. F. Quinn has shown that this homomorphism is in fact an isomorphism [Q].

Now assume in addition that $X$ is a differentiable manifold. Then one considers the component group $\text{Diff}_+(X)$ of the group of orientation preserving diffeomorphisms of $X$, and again there is a natural homomorphism $\psi: \text{Diff}_+(X) \to O(L)$. There are some classical results related to $\psi$ which have been obtained via traditional (surgery) methods: It was first observed by C.T.C. Wall [W] that $\psi$ is surjective for $S^2 \times S^2$, for the connected sum $\mathbf{P}^2 \# \overline{\mathbf{P}^2}$ of the complex projective plane $\mathbf{P}^2$ with another copy $\overline{\mathbf{P}^2}$ with reversed orientation, and for the connected sum of several copies of $\mathbf{P}^2$. Wall has also shown that $\psi$ is surjective if $X$ is of the form $X = Y \# S^2 \times S^2$ where $Y$ is a manifold whose quadratic form $q_Y$ is indefinite or has rank $\leqslant 8$ [W]. M. Kreck has proved that the image of $\psi$ is isomorphic to the group $\widetilde{\mathrm{Diff}}_+(X)$ of pseudo-isotopy classes of orientation preserving diffeomorphisms [Kr].

In general it is therefore a difficult problem to determine the image of $\psi$.

With the new methods provided by Donaldson, the following results were obtained. Suppose that $X$ is a complex algebraic surface with canonical class $k_X$. For a Dolgachev surface $X$, R. Friedman and J. Morgan have shown that the image of $\psi$ is of finite index in the subgroup of $O(L)$ consisting of isometries preserving $\{\pm k_X\}$; this subgroup itself is of infinite index in $O(L)$ [FM]. Recently Donaldson has determined the image of $\psi$ for a K3 surface $X$; he has identified it with a certain subgroup of index 2 in $O(L)$ [D].

The main result of this paper describes the image of $\psi$ for other types of algebraic surfaces. For the precise formulation of the result we refer to §4. The proof is inspired by Donaldson's proof. It has three main parts.

First we exhibit a large subgroup of $\psi(\mathrm{Diff}_+(X))$. This is the monodromy group of a smooth family of surfaces containing $X$ as a fibre. In many cases we can determine the monodromy group of a suitable family. Moreover a diffeomorphism can be constructed using complex conjugation. §1 deals with these topics.

The next ingredient is the $C^\infty$-invariance of the canonical class. This is proved for certain algebraic surfaces using Donaldson's $SO(3)$-polynomial invariants and the results of [FMM]. This is the subject of §2.

Finally we show (in §3) that $-1 \in O(L)$ is not induced by an orientation preserving diffeomorphism if $X$ is an algebraic surface with odd geometric genus.

The main new ingredient is the use of Donaldson's $SO(3)$-invariants, where we rely on the recent nontriviality result of K. Zuo [Z].

We conclude the paper with two applications. The first one concerns the problem of representing homology classes in $H_2(X, \mathbf{Z})$ by differentiably embedded 2-spheres. For a second application we consider homeomorphic surfaces $X$ and $X'$ to which our results apply and for which we can determine the images of the corresponding homomorphisms $\psi$. We show that if the divisibilities of the canonical classes of $X$ and $X'$ in integral cohomology are different (and hence the surfaces are not diffeomorphic), then the corresponding images are non-conjugate subgroups of $O(L)$, and vice versa. Finally we give an example of two such surfaces.

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## 1. CONSTRUCTION OF DIFFEOMORPHISMS

Let $X$ be a simply connected smooth compact complex algebraic surface with quadratic intersection form $q_X : H_2(X, \mathbf{Z}) \to \mathbf{Z}$ and canonical class $k_X \in H^2(X, \mathbf{Z}) = \mathrm{Hom}(H_2(X, \mathbf{Z}), \mathbf{Z})$. The second homology group $H_2(X, \mathbf{Z})$ endowed with the quadratic form $q_X$ forms a lattice which we denote by $L$. Let $O(L)$ be the corresponding group of isometries.

We introduce some notation: Let $b_2^+(X)$ denote the dimension of any maximal subspace of $L_{\mathbf{R}} = L \otimes \mathbf{R}$ on which $q_X$ is positive definite. Note that $b_2^+(X) = 2p_g(X) + 1$ where $p_g(X)$ is the geometric genus of $X$. The set of all oriented maximal positive definite subspaces of $L_{\mathbf{R}}$ forms an open subset $\Omega$ of the Grassmannian $G^{\mathrm{or}}(b_2^+(X), L_{\mathbf{R}})$ of oriented $b_2^+(X)$-dimensional subspaces of the vector space $L_{\mathbf{R}}$. It has two components if $q_X$ is indefinite. We define $O'(L)$ to be the subgroup of $O(L)$ consisting of those automorphisms which leave each component of $\Omega$ invariant. (For an equivalent definition of $O'(L)$ see [E2, 4.1].) Let $O_k(L)$ be the subgroup of $O(L)$ consisting of automorphisms preserving $k = k_X$. Finally we define $O'_k(L) := O_k(L) \cap O'(L)$.

An important subset of $\mathrm{Diff}_+(X)$ is the set of classes of diffeomorphisms obtained by monodromy transformations of a smooth family containing $X$ as a fibre. By a *smooth family* we mean a smooth (in the analytic category) proper holomorphic mapping $\pi : \mathscr{X} \to T$ of connected complex spaces $\mathscr{X}$ and $T$; $\pi$ is the projection of a locally trivial differentiable fibre bundle, so that for a point $t_0 \in T$ with $X = \pi^{-1}(t_0)$ there is a *monodromy representation* $\rho : \pi_1(T, t_0) \to \mathrm{Diff}_+(X)$. The image $\Gamma$ of $\psi \circ \rho$ in $O(L)$ is called the *monodromy group* of the smooth family. The monodromy group preserves $k_X$. It also preserves the components of $\Omega$: to see this consider a loop $\tau$ representing an element in $\pi_1(T, t_0)$ and let $g_t : X_{t_0} = X \to X_t$ be the diffeomorphisms corresponding to $t \in \tau$. The mapping $t \mapsto (g_t)_*(\alpha) \in G^{\mathrm{or}}(b_2^+(X), L_{\mathbf{R}})$ is continuous for every $\alpha \in \Omega$, hence $\Gamma \subset O'_k(L)$.

For certain algebraic surfaces there exist smooth families whose monodromy group is the whole group $O'_k(L)$. This is summarized in the following theorem. Among these surfaces there are certain iterated ramified coverings of $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$ which were studied by Moishezon [M] and Salvetti [S]: Let $(n_1, \ldots, n_r)$ be a sequence of positive integers. For Moishezon's construction, let $X_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and let $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the divisor $\{\mathrm{pt}\} \times \mathbf{P}^1 + \mathbf{P}^1 \times \{\mathrm{pt}\}$. Construct a sequence $\beta_i : X_i \to X_{i-1}$ of cyclic coverings $\beta_i$ of degree 3 with nonsingular branch locus linearly equivalent to $(\beta_{i-1} \circ \ldots \circ \beta_1)^*(3n_i C)$. Let $X_r = X_r(n_1, \ldots, n_r)$. We call a surface $X_r(n_1, \ldots, n_r)$ with $n_i \geqslant 2$ for some $i$, $1 \leqslant i \leqslant r$, (cf. [M, §4, Remark 3]) a *Moishezon surface.* For Salvetti's construction we need in addition to $(n_1, \ldots, n_r)$ a sequence $(d_1, \ldots, d_r)$ of positive integers satisfying $d_i \mid n_i$ for all $i = 1, \ldots, r$. Let $Y_0 = \mathbf{P}^2$, and choose smooth curves $C_i \subset \mathbf{P}^2$ of degree $n_i$, $i = 1, \ldots, r$, so that $C := C_1 \cup \ldots \cup C_r$ has only normal crossings. Construct a sequence $\beta_i : Y_i \to Y_{i-1}$ of cyclic coverings $\beta_i$ of degree $d_i$ ramified over $(\beta_{i-1} \circ \ldots \circ \beta_1)^*(C_i)$. A *Salvetti surface* is a surface $Y_r = Y_r(n_1, \ldots, n_r; d_1, \ldots, d_r)$, if at least one $n_i > 5$ with the corresponding $d_i \geqslant 3$ (cf. [S, Corollary to Proposition 2]).

THEOREM 1. *Let $X$ be a simply connected algebraic surface with $p_g(X) > 0$, and suppose that $X$ is either*

(i) *a complete intersection, or*

(ii) *a Moishezon or Salvetti surface.*

*Then there is a smooth family $\pi : \mathscr{X} \to T$ with $\pi^{-1}(t_0) = X$ for some $t_0 \in T$ with monodromy group*

$$\Gamma = O'_k(L) .$$

For complete intersection surfaces this is proved in [B, E2, E3], for Moishezon surfaces see [M], and for Salvetti surfaces see [S], together with [EO, Theorem 2.5 and §1].

Next we construct an orientation preserving diffeomorphism $\sigma : X \to X$ with $\sigma^*(k_X) = -k_X$. Suppose $X$ is embedded in a complex projective space $\mathbf{P}^N$ so that $X$ is the zero locus of a finite set $\{f_1, \ldots, f_M\}$ of homogeneous polynomials in the coordinates $z_0, \ldots, z_N$ of $\mathbf{P}^N$.

Denote by $\sigma : \mathbf{C} \to \mathbf{C}$ complex conjugation, and let $X^\sigma \subset \mathbf{P}^N$ be the zero locus of $\{f_1^\sigma, \ldots, f_M^\sigma\}$, where $f_i^\sigma$ is obtained from $f_i$ by applying $\sigma$ to all the coefficients. Then $\sigma : \mathbf{P}^N \to \mathbf{P}^N$ induces a diffeomorphism $X \to X^\sigma$, also denoted by $\sigma$, which satisfies $\sigma^*(k_{X^\sigma}) = -k_X$. If $X$ is given by real equations $f_1, \ldots, f_M$, then $\sigma : X \to X^\sigma = X$ is a self-diffeomorphism of $X$. Such equations can be found if $X$ is a complete intersection and if $X$ is a

Moishezon or Salvetti surface. (In the latter case the branch locus must be given by real equations.)

Therefore we have:

COROLLARY 2. *Let $X$ be an algebraic surface as in Theorem 1. Then*

$$O'_k(L) \cdot \{\sigma_*, \mathrm{id}\} \subset \psi(\mathrm{Diff}_+(X)) .$$

## 2. INVARIANCE OF THE CANONICAL CLASS

S. K. Donaldson [D] has defined a series of invariants for certain smooth 4-manifolds. They are in particular defined for simply connected algebraic surfaces $X$ with $p_g(X) > 0$. We assume from now on that $X$ is such a surface. There are two types of invariants according to the gauge group being $SU(2)$ or $SO(3)$.

Let us first recall the $SU(2)$-case. Principal $SU(2)$-bundles over $X$ are classified by their second Chern class $c_2(P)$. For each $l > l_0$, using such a bundle with $c_2(P) = l$, Donaldson defines a polynomial

$$\Phi_l(X): \mathrm{Sym}^d(L) \to \mathbf{Z}$$

of degree $d = d(l) = 4l - 3(p_g(X) + 1)$, which depends only on the underlying $C^\infty$-structure of $X$ and is invariant up to sign under $\psi(\mathrm{Diff}_+(X))$. Donaldson shows that these invariants are nontrivial for all sufficiently large $l$ [D].

We will need the slightly more complicated $SO(3)$-invariants. The simple Lie group $SO(3)$ is isomorphic to $PU(2)$, so that one has an exact sequence

$$1 \to S^1 \to U(2) \to SO(3) \to 1 .$$

Let $P$ be a principal $SO(3)$-bundle over $X$. Such a bundle has two characteristic classes which determine it up to isomorphism: the second Stiefel-Whitney class $w_2(P) \in H^2(X, \mathbf{Z}/2)$ and the first Pontryagin class $p_1(P) \in H^4(X, \mathbf{Z})$.

Suppose that $w_2(P)$ is nonzero and choose an integral lifting $c$ of $w_2(P)$, i.e. $c \in H^2(X, \mathbf{Z})$, $\bar{c} = w_2(P)$ (here $\bar{c}$ means the reduction of $c$ modulo 2). Such a lifting exists since $X$ is simply connected, and determines a $U(2)$-lifting $\hat{P}$ of $P$, i.e. a $U(2)$-bundle $\hat{P}$ with $\hat{P}/S^1 = P$ and with $c = c_1(\hat{P})$ [HH]. The Chern classes of $\hat{P}$ are related to the characteristic classes of $P$ by $w_2(P) = \bar{c}_1(\hat{P})$ and $p_1(P) = c_1^2(\hat{P}) - 4c_2(\hat{P})$. In addition to this choose an element $\alpha \in \Omega$. Donaldson shows that these choices give rise to a polynomial

$$\Phi_{c,\alpha,P}(X): \mathrm{Sym}^d(L) \to \mathbf{Z}$$

of degree $d = -p_1(P) - 3(p_g(X)+1) = 4c_2(\hat{P}) - c^2 - 3(p_g(X)+1)$ with the following properties ([D], see also [OV]):

(a) $\Phi_{c,-\alpha,P}(X) = -\Phi_{c,\alpha,P}(X)$ where $-\alpha$ is the subspace corresponding to $\alpha$ with the opposite orientation.

(b) $\Phi_{c+2a,\alpha,P}(X) = \varepsilon(a)\Phi_{c,\alpha,P}(X)$ where

$$\varepsilon(a) = \begin{cases} 1 & \text{if} \quad \bar{a}^2 = 0\,, \\ -1 & \text{if} \quad \bar{a}^2 \neq 0\,. \end{cases}$$

(c) If $f: X' \to X$ is an orientation preserving diffeomorphism then

$$\Phi_{f^*(c),f^*(\alpha),f^*(P)}(X') = f^*\Phi_{c,\alpha,P}(X)\,.$$

Donaldson's nontriviality result for the $SU(2)$-invariants has been extended to the $SO(3)$-case by Zuo [Z]:

THEOREM 3 (Zuo). *Let $X$ be a simply connected algebraic surface with $p_g(X) > 0$. If $c \in H^{1,1}(X, \mathbf{Z})$, $\bar{c} \neq 0$, and $P$ is a principal $SO(3)$-bundle corresponding to a $U(2)$-bundle $\hat{P}$ with $c_1(\hat{P}) = c$ and $c_2(\hat{P})$ sufficiently large, then the polynomial $\Phi_{c,\alpha,P}(X)$ is nontrivial.*

Now suppose that $X$ has a big monodromy group in the sense of Friedman and Morgan [FMM]. Then the $SU(2)$-invariants $\Phi_l(X)$ of $X$ are complex polynomials in the canonical class $k_X$ and the quadratic form $q_X$ [FMM]. In the $SO(3)$-case one finds the following result:

THEOREM 4. *Let $X$ be a simply connected algebraic surface with $p_g(X) > 0$, $w_2(X) \neq 0$, and with a big monodromy group. Then, for a principal $SO(3)$-bundle $P$,*

$$\Phi_{k_X,\alpha,P}(X) \in \mathbf{C}[k_X, q_X]\,.$$

COROLLARY 5. *Let $X$ be a simply connected algebraic surface with $p_g(X) > 0$ and with a big monodromy group. Then $\{\pm k_X\}$ is invariant under $\psi(\mathrm{Diff}_+(X))$, if $k_X$ divides a nontrivial polynomial invariant.*

The corollary follows from the fact that if $k_X$ divides a nontrivial polynomial invariant, then it is its only linear factor up to multiples (cf. [FMM]).

When are the assumptions of Corollary 5 satisfied? It follows from Theorem 1 that the surfaces listed in this theorem have big monodromy.

Let $X$ be any simply connected algebraic surface with a big monodromy group. If $p_g(X) \equiv 0 \pmod 2$ then the degree of $\Phi_l(X)$ is odd. If $p_g(X) \equiv 1 \pmod 2$ and $k_X^2 \equiv 1 \pmod 2$ then the degree of $\Phi_{k_X,\alpha,P}(X)$ is odd. So $k_X$ divides $\Phi_l(X)$ or $\Phi_{k_X,\alpha,P}(X)$ in these cases.

*Remark.* Theorem 4 and its corollary remain true for polynomials $\Phi_{c,\alpha,P}(X)$ if $c \in H^2(X, \mathbf{Z})$ is a class with $\bar{c} \neq 0$ such that $\overline{f^*(c)} = \bar{c}$ for all $f \in \psi(\mathrm{Diff}_+(X))$. The question which elements of $H^2(X, \mathbf{Z})$ or $H^2(X, \mathbf{Z}/2)$ have this invariance property will be treated in §4.

## 3. Non-realizable isometries

We shall show that for a simply connected algebraic surface with odd geometric genus, $-1$ is not induced by an orientation preserving diffeomorphism. For K3 surfaces this was shown by Donaldson in the proof of [D, Proposition 6.2]. There he proves the nontriviality of a certain polynomial $\Phi_{c,\alpha,P}(X)$ for a K3 surface $X$. With Zuo's nontriviality result (Theorem 3) we are able to generalize this as follows.

Theorem 6. *If $X$ is a simply connected algebraic surface with $p_g(X) \equiv 1 \pmod 2$ then $-1 \notin \psi(\mathrm{Diff}_+(X))$.*

*Proof.* Suppose that there is an orientation preserving diffeomorphism $f: X \to X$ such that $f^* = -1$. Let $c \in H^{1,1}(X, \mathbf{Z})$ be a class with $\bar{c} \neq 0$, and choose a principal $SO(3)$-bundle $P$ with $w_2(P) = \bar{c}$ such that $\Phi_{c,\alpha,P}(X)$ is nontrivial. This is possible according to Theorem 3. Then

$$f^*\Phi_{c,\alpha,P}(X) = (-1)^d \Phi_{c,\alpha,P}(X) ,$$

since $\Phi_{c,\alpha,P}(X)$ is a polynomial of degree $d$ on $L$.

On the other hand, by §2(c)

$$f^*\Phi_{c,\alpha,P}(X) = \Phi_{f^*c, f^*\alpha, f^*P}(X) .$$

We have $f^*c = -c$ and $f^*\alpha = -\alpha$ because $f^* = -1$ and the dimension of $\alpha$ is odd. Since $f$ is orientation preserving and $f^* = -1$ we find $f^*p_1(P) = p_1(P)$ and $f^*w_2(P) = w_2(P)$, so that the bundle $f^*P$ is isomorphic to $P$. Therefore

$$f^* \Phi_{c,\alpha,P}(X) = \Phi_{-c,-\alpha,P}(X) .$$

Applying §2(a) and (b) with $a = -c$ we get

$$f^* \Phi_{c,\alpha,P}(X) = -\Phi_{-c,\alpha,P}(X) = -(-1)^{c^2} \Phi_{c,\alpha,P}(X) .$$

By assumption $\Phi_{c,\alpha,P}(X) \not\equiv 0$, so we must have

$$(-1)^{c^2+1} = (-1)^d .$$

Now $d = 4c_2 - c^2 - 3(1 + p_g(X))$ implies that $(-1)^{p_g(X)} = 1$, i.e. $p_g(X) \equiv 0 \pmod 2$. This proves Theorem 6.

## 4. Diffeomorphism groups of some algebraic surfaces

In §1 we saw that the image of $\psi$ contains the group $O_k'(L) \cdot \{\sigma_*, \mathrm{id}\}$ in many cases. In §2 we showed that under certain conditions $\{\pm k_X\}$ is invariant under $\psi(\mathrm{Diff}_+(X))$. Finally we proved in the previous section that for algebraic surfaces of odd geometric genus $-1$ is not induced by an orientation preserving diffeomorphism. It turns out that these facts suffice to determine the image of $\psi$.

PROPOSITION 7. *Let $X$ be a simply connected algebraic surface which satisfies the following conditions:*

(i) $O_k'(L) \cdot \{\sigma_*, \mathrm{id}\} \subset \psi(\mathrm{Diff}_+(X))$,

(ii) $\{\pm k_X\}$ *is invariant under* $\psi(\mathrm{Diff}_+(X))$,

(iii) $-1 \notin \psi(\mathrm{Diff}_+(X))$.

*Then*

$$\psi(\mathrm{Diff}_+(X)) = O_k'(L) \cdot \{\sigma_*, \mathrm{id}\} .$$

*Proof.* Let $g = -\sigma_*$. Then $g \in O_k(L)$, but $g \notin \psi(\mathrm{Diff}_+(X))$, since $-1 \notin \psi(\mathrm{Diff}_+(X))$. Hence by (i), $g \notin O_k'(L)$. Therefore

$$O_k(L) = O_k'(L) \cdot \{g, \mathrm{id}\} .$$

Now let $h \in \psi(\mathrm{Diff}_+(X))$. By (ii) either $h(k) = k$ or $h(k) = -k$. In the first case $h \in O_k(L)$. Moreover, $h \in O_k'(L)$ since otherwise $h = gh_0$ for some $h_0 \in O_k'(L)$ which would imply $g \in \psi(\mathrm{Diff}_+(X))$, a contradiction. In the second case we have $h' = h\sigma_* \in O_k(L)$. By the same argument as before we see that $h' \in O_k'(L)$. Hence $h = h'\sigma_* \in O_k'(L) \cdot \{\sigma_*, \mathrm{id}\}$. This proves Proposition 7.

Putting everything together we get the main result of our paper.

THEOREM 8. *Let $X$ be a simply connected algebraic surface with $p_g(X) \equiv 1 \pmod 2$ and $k_X^2 \equiv 1 \pmod 2$. If $X$ is either*

(i) *a complete intersection, or*

(ii) *a Moishezon or Salvetti surface,*

*then*

$$\psi(\mathrm{Diff}_+(X)) = O'_k(L) \cdot \{\sigma_*, \mathrm{id}\} .$$

*Example.* Consider a complete intersection surface $X$ of multidegree $(d_1, \ldots, d_r)$. Using the formulas of [E4], we can translate the conditions of Theorem 8 into numerical conditions on the degrees $d_i$. The condition $k_X^2 \equiv 1 \pmod 2$ is equivalent to

$$(1) \qquad d_i \equiv 1 \pmod 2 \quad \text{for} \quad i = 1, \ldots, r .$$

Write $d_i = 2e_i + 1$ $(i = 1, \ldots, r)$. Then $p_g(X) \equiv 1 \pmod 2$ is equivalent to the following two conditions

$$(2) \qquad \sum_{i<j} e_i e_j \equiv 1 \pmod 2 ,$$

$$(3) \qquad \text{either } 3 \mid d_j \text{ for some } j,\ 1 \leqslant j \leqslant r, \quad \text{or} \quad 3 \mid \textstyle\sum_{i=1}^{r} e_i(e_i + 1) .$$

In particular there is an infinite sequence of complete intersection surfaces satisfying the conditions of Theorem 8, e.g. the surfaces with $(d_1, d_2) = (3, 3 + 4m)$, $m \in \mathbf{Z}$, $m \geqslant 0$.

We leave it to the reader to formulate similar conditions for the case (ii) of Theorem 8.

We shall give two further applications of the results of the first three sections.

Let $X$ be a surface as in Theorem 1 and denote the symmetric bilinear form corresponding to $q_X$ by $\langle\ ,\ \rangle$. Define $L' := \ker k_X = k_X^{\perp} \subset L$, and let $\Delta \subset L'$ be the set of vanishing cycles of $X$ (cf. [EO]). The pair $(L', \Delta)$ is then a vanishing lattice in the sense of [E1, Definition (2.1)]. This means the following. If $\delta \in \Delta$, then $\langle \delta, \delta \rangle = -2$ and one has an associated reflection $s_\delta$ defined by

$$s_\delta(x) = x + \langle x, \delta \rangle \delta$$

for all $x \in L'$. Let $\Gamma_\Delta$ denote the subgroup of $O(L')$ generated by these reflections $s_\delta$, $\delta \in \Delta$. Then $(L', \Delta)$ satisfies the following conditions:

(i) $\langle \delta, \delta \rangle = -2$ for all $\delta \in \Delta$.

(ii) $\Delta$ generates $L'$.

(iii) $\Delta$ is a $\Gamma_\Delta$-orbit.

(iv) Unless rank $L' = 1$, there exist $\delta_1, \delta_2 \in \Delta$ with $\langle \delta_1, \delta_2 \rangle = 1$.

As in Wall's paper [W] we can derive from a statement about $\psi(\mathrm{Diff}_+(X))$ a statement about the possibility of representing homology classes by embedded 2-spheres.

THEOREM 9. *Let* $X$ *be an algebraic surface, and let* $x \in H_2(X, \mathbf{Z})$ *be a class with* $q_X(x) = -2$. *If* $x$ *is represented by a differentiably embedded 2-sphere, then* $\bar{x} \in \bar{k}_X^\perp$. *Conversely, if* $X$ *is a surface as in Theorem 1, if* $x \in k_X^\perp$, *and if there exists a class* $y \in k_X^\perp$ *with* $\langle x, y \rangle = 1$, *then* $x$ *can be represented by a differentiably embedded 2-sphere.*

*Proof.* Let $x \in H_2(X, \mathbf{Z})$ be a class with $q_X(x) = -2$. Suppose that $x$ is represented by a differentiably embedded 2-sphere $S$. Let $j: S \to X$ be the embedding. The normal bundle $N_S$ of $S$ in $X$ can be regarded as a $U(1)$-bundle. Therefore the first Chern class $c_1(N_S)$ of the normal bundle is defined. Let $\xi \in H^2(X, \mathbf{Z})$ be the Poincaré dual of $x$. Then by [H, Theorem 4.8.1] $c_1(N_S) = j^*\xi$. If $T_X$ and $T_S$ denote the tangent bundles of $X$ and $S$ respectively, then we have $j^*\bar{k}_X = w_2(j^*T_X) = w_2(T_S) + w_2(N_S)$. This implies $\langle \bar{k}_X, \bar{\xi} \rangle = \overline{\chi(S) + \xi^2} = 0$ where $\chi(S)$ denotes the Euler characteristic of $S$. It follows that $\bar{x} \in \bar{k}_X^\perp$.

Conversely, let $X$ be a surface as in Theorem 1. Then $(L', \Delta)$ is a complete vanishing lattice in the sense of [E1, Definition (2.2)]. This follows for complete intersection surfaces from [B, E3], for Moishezon surfaces from [M], and for Salvetti surfaces from [S] (see also [EO]). By [E1, Proposition (2.5)] we conclude that

$$\Delta = \{v \in L' \mid q_X(v) = -2 \quad \text{and} \quad \langle v, L' \rangle = \mathbf{Z}\}\,.$$

Therefore, if $x \in L'$ and if there exists a $y \in L'$ with $\langle x, y \rangle = 1$, then $x \in \Delta$, i.e. $x$ is a vanishing cycle. But vanishing cycles are certainly represented by spheres. This proves Theorem 9.

*Remark.* We have even proved more, namely that every $x$ satisfying the latter conditions of Theorem 9 is a vanishing cycle.

Our second application concerns a question which was posed by E. Brieskorn. First we show:

PROPOSITION 10. *Let $X$ be an algebraic surface as in Theorem 1. Let $x \in \mathrm{Hom}(L, \mathbf{Z})$. If $\{\pm x\}$ is invariant under $\psi(\mathrm{Diff}_+(X))$, then $x = \lambda k_X$ for some $\lambda \in \mathbf{Q}$, unless $k_X^2 = 0, k_X \neq 0$.*

*Proof.* Let $U \subset L'_{\mathbf{Q}} = L' \otimes \mathbf{Q}$ be a subspace of $L'_{\mathbf{Q}}$ which is invariant under $\psi(\mathrm{Diff}_+(X))$. We show that either $U = L'_{\mathbf{Q}}$ or $U$ is contained in $(L'_{\mathbf{Q}})^{\perp} = \{v \in L'_{\mathbf{Q}} \mid \langle v, w \rangle = 0 \text{ for all } w \in L'_{\mathbf{Q}}\}$. Let $\Delta \subset L'$ be the set of vanishing cycles. If $\delta \in \Delta$ is not orthogonal to $U$ then there exists a $y \in U$ with $\langle \delta, y \rangle \neq 0$. From $s_\delta(y) = y + \langle y, \delta \rangle \delta \in U$ it follows that $\delta \in U$. Since $\Gamma_\Delta$ acts transitively on $\Delta$, we obtain $\Delta \subset U$. But $\Delta$ generates $L'$ so that we must have $L'_{\mathbf{Q}} = U$.

Now let $x \in \mathrm{Hom}(L, \mathbf{Z})$ be invariant up to sign under $\Gamma_\Delta \subset \psi(\mathrm{Diff}_+(X))$. If $k_X = 0$ then $L = L'$ and $(L')^{\perp} = \{0\}$. Hence it follows from what we have just shown that $x = 0$. If $k_X^2 \neq 0$ then we can write $x = \lambda k_X + x_0$ where $\lambda \in \mathbf{Q}$ and $x_0 \in \mathrm{Hom}(L', \mathbf{Z}) \otimes \mathbf{Q}$. Now $k_X$ is invariant under $\Gamma_\Delta$, so we see that $\{\pm x_0\}$ is invariant under $\Gamma_\Delta$. Since $(L')^{\perp} = \{0\}$, it follows that $x_0 = 0$. This proves Proposition 10.

Now $(\overline{L'}, \bar{\Delta})$, where $\bar{\Delta} = \{\bar{\delta} \mid \delta \in \Delta\}$, is also a vanishing lattice. So we can derive the following proposition by the same arguments.

PROPOSITION 11. *Let $X$ be an algebraic surface as in Theorem 1. Write $k_X = n_X \kappa_X$ for some primitive element $\kappa_X \in \mathrm{Hom}(L, \mathbf{Z})$ and some non-negative integer $n_X$. If $x \in \mathrm{Hom}(L, \mathbf{Z})$ is an element with $\overline{g(x)} = \bar{x}$ for all $g \in \psi(\mathrm{Diff}_+(X))$, then $\bar{x} \in \{0, \overline{\kappa_X}\}$, unless $\kappa_X^2 = 0, \kappa_X \neq 0$.*

Consider two simply connected algebraic surfaces $X$ and $X'$ with corresponding lattices $L_X$ and $L_{X'}$. Let $h: X \to X'$ be an orientation preserving homeomorphism between $X$ and $X'$. Then $h_*: L_X \to L_{X'}$ is an isometry. For a subgroup $G \subset O(L_{X'})$ we define $G^{h_*} := h_*^{-1} G h_*$. Note that we have $k_X^2 = k_{X'}^2$, since $k_X^2$ can be expressed in terms of the rank and signature of $L_X$.

THEOREM 12. *Let $X, X'$ be algebraic surfaces as in Theorem 8, and suppose that $k_X^2 = k_{X'}^2 \neq 0$. Let $h: X \to X'$ be an orientation preserving homeomorphism. Then $\psi(\mathrm{Diff}_+(X))$ and $\psi(\mathrm{Diff}_+(X'))^{h_*}$ are conjugate subgroups in $O(L_X)$ if and only if the divisibilities of $k_X$ and $k_{X'}$ in integral cohomology are equal.*

*Proof.* Suppose that $\psi(\mathrm{Diff}_+(X))$ and $\psi(\mathrm{Diff}_+(X'))^{h_*}$ are conjugate by $g \in O(L_X)$, i.e. $\psi(\mathrm{Diff}_+(X)) = (\psi(\mathrm{Diff}_+(X'))^{h_*})^g$. Let $f \in \psi(\mathrm{Diff}_+(X'))$. Then

$$(h_*g)^{-1}f(h_*g)(k_X) = \pm k_X ,$$

since $\{\pm k_X\}$ is invariant under $\psi(\mathrm{Diff}_+(X))$. Therefore

$$f((h_*g)(k_X)) = \pm (h_*g)(k_X) .$$

This holds for every $f \in \psi(\mathrm{Diff}_+(X'))$, so that $\{\pm (h_*g)(k_X)\}$ is invariant under $\psi(\mathrm{Diff}_+(X'))$. We conclude from Proposition 10 that $(h_*g)(k_X) = \lambda k_{X'}$ for some $\lambda \in \mathbf{Q}$. Now

$$k_{X'}^2 = k_X^2 = ((h_*g)(k_X))^2 = \lambda^2 k_{X'}^2 ,$$

hence $\lambda = \pm 1$, and it follows that the divisibilities of $k_X$ and $k_{X'}$ in integral cohomology are equal.

Conversely, let $k_X = n_X \kappa_X$ and $k_{X'} = n_{X'}\kappa_{X'}$ for some primitive elements $\kappa_X \in H^2(X, \mathbf{Z})$ and $\kappa_{X'} \in H^2(X', \mathbf{Z})$ and some nonnegative integers $n_X$ and $n_{X'}$ respectively, and assume that $n_X$ and $n_{X'}$ are equal. Let $\lambda_X$ and $\lambda_{X'}$ be the Poincaré duals of $\kappa_X$ and $\kappa_{X'}$, and let $K$ and $K'$ be the one-dimensional sublattices of $L_X$ spanned by $\lambda_X$ and $h_*^{-1}(\lambda_{X'})$ respectively. Let $g: K \to K'$ be the homomorphism defined by $g(\lambda_X) = h_*^{-1}(\lambda_{X'})$. Then $\langle g(\lambda_X), g(\lambda_X)\rangle = \langle \lambda_X, \lambda_X \rangle$, hence $g$ is an isometry. Now $b_2^+(X) \geqslant 3$, so that by a generalization of Witt's theorem [N, § 1.14, in particular 1.14.4, § 1.16] $g$ can be extended to an isometry $g \in O(L_X)$. One can easily verify that $\psi(\mathrm{Diff}_+(X))$ and $\psi(\mathrm{Diff}_+X'))^{h_*}$ are conjugate by $g$. This proves Theorem 12.

COROLLARY 13. *Let* $X$ *be an algebraic surface as in Theorem 8. If an element* $h \in O'(L)$ *normalizes* $\psi(\mathrm{Diff}_+(X))$, *then* $h$ *is contained in* $\psi(\mathrm{Diff}_+(X))$.

*Proof.* This follows from Theorem 12, because

$$\psi(\mathrm{Diff}_+(X)) = O_k'(L) \cdot \{\sigma_*, \mathrm{id}\} .$$

*Remark.* Since $-1$ is not contained in $\psi(\mathrm{Diff}_+(X))$ but in the normalizer $\mathrm{Norm}(\psi(\mathrm{Diff}_+(X)))$ of $\psi(\mathrm{Diff}_+(X))$ we obtain from Corollary 13:

$$\mathrm{Norm}(\psi(\mathrm{Diff}_+(X)))/\psi(\mathrm{Diff}_+(X)) \cong \mathbf{Z}/2 \cong \{\pm \mathrm{id}\} .$$

*Example.* Let $X$ be a complete intersection in $\mathbf{P}^6$ of multidegree $(7,7,5,3)$. Let $X'$ be the Salvetti surface $Y_4(10,10,6,5;5,5,3,5)$. Both surfaces have $k^2 = 165375$, $p_g = 24499$, but the divisibilities of the canonical classes are 15 and 21 respectively. Therefore the surfaces are homeomorphic, but not diffeomorphic. Both surfaces satisfy the assumptions of Theorem 8.

We conclude from Theorem 12 that the subgroups of $O(L)$ corresponding to $\psi(\mathrm{Diff}_+(X))$ and $\psi(\mathrm{Diff}_+(X'))$ are not conjugate. This example was found by a computer search.

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