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Moishezon or Salvetti surface. (In the latter case the branch locus must be given by real equations.)

Therefore we have:

COROLLARY 2. *Let  $X$  be an algebraic surface as in Theorem 1. Then*

$$O'_k(L) \cdot \{\sigma_*, \text{id}\} \subset \psi(\text{Diff}_+(X)) .$$

## 2. INVARIANCE OF THE CANONICAL CLASS

S. K. Donaldson [D] has defined a series of invariants for certain smooth 4-manifolds. They are in particular defined for simply connected algebraic surfaces  $X$  with  $p_g(X) > 0$ . We assume from now on that  $X$  is such a surface. There are two types of invariants according to the gauge group being  $SU(2)$  or  $SO(3)$ .

Let us first recall the  $SU(2)$ -case. Principal  $SU(2)$ -bundles over  $X$  are classified by their second Chern class  $c_2(P)$ . For each  $l > l_0$ , using such a bundle with  $c_2(P) = l$ , Donaldson defines a polynomial

$$\Phi_l(X): \text{Sym}^d(L) \rightarrow \mathbf{Z}$$

of degree  $d = d(l) = 4l - 3(p_g(X) + 1)$ , which depends only on the underlying  $C^\infty$ -structure of  $X$  and is invariant up to sign under  $\psi(\text{Diff}_+(X))$ . Donaldson shows that these invariants are nontrivial for all sufficiently large  $l$  [D].

We will need the slightly more complicated  $SO(3)$ -invariants. The simple Lie group  $SO(3)$  is isomorphic to  $PU(2)$ , so that one has an exact sequence

$$1 \rightarrow S^1 \rightarrow U(2) \rightarrow SO(3) \rightarrow 1 .$$

Let  $P$  be a principal  $SO(3)$ -bundle over  $X$ . Such a bundle has two characteristic classes which determine it up to isomorphism: the second Stiefel-Whitney class  $w_2(P) \in H^2(X, \mathbf{Z}/2)$  and the first Pontryagin class  $p_1(P) \in H^4(X, \mathbf{Z})$ .

Suppose that  $w_2(P)$  is nonzero and choose an integral lifting  $c$  of  $w_2(P)$ , i.e.  $c \in H^2(X, \mathbf{Z})$ ,  $\bar{c} = w_2(P)$  (here  $\bar{c}$  means the reduction of  $c$  modulo 2). Such a lifting exists since  $X$  is simply connected, and determines a  $U(2)$ -lifting  $\hat{P}$  of  $P$ , i.e. a  $U(2)$ -bundle  $\hat{P}$  with  $\hat{P}/S^1 = P$  and with  $c = c_1(\hat{P})$  [HH]. The Chern classes of  $\hat{P}$  are related to the characteristic classes of  $P$  by  $w_2(P) = \bar{c}_1(\hat{P})$  and  $p_1(P) = c_1^2(\hat{P}) - 4c_2(\hat{P})$ . In addition to this choose an element  $\alpha \in \Omega$ . Donaldson shows that these choices give rise to a polynomial

$$\Phi_{c, \alpha, P}(X): \text{Sym}^d(L) \rightarrow \mathbf{Z}$$

of degree  $d = -p_1(P) - 3(p_g(X) + 1) = 4c_2(\hat{P}) - c^2 - 3(p_g(X) + 1)$  with the following properties ([D], see also [OV]):

(a)  $\Phi_{c, -\alpha, P}(X) = -\Phi_{c, \alpha, P}(X)$  where  $-\alpha$  is the subspace corresponding to  $\alpha$  with the opposite orientation.

(b)  $\Phi_{c+2a, \alpha, P}(X) = \varepsilon(a)\Phi_{c, \alpha, P}(X)$  where

$$\varepsilon(a) = \begin{cases} 1 & \text{if } \bar{a}^2 = 0, \\ -1 & \text{if } \bar{a}^2 \neq 0. \end{cases}$$

(c) If  $f: X' \rightarrow X$  is an orientation preserving diffeomorphism then

$$\Phi_{f^*(c), f^*(\alpha), f^*(P)}(X') = f^*\Phi_{c, \alpha, P}(X).$$

Donaldson's nontriviality result for the  $SU(2)$ -invariants has been extended to the  $SO(3)$ -case by Zuo [Z]:

**THEOREM 3 (Zuo).** *Let  $X$  be a simply connected algebraic surface with  $p_g(X) > 0$ . If  $c \in H^{1,1}(X, \mathbf{Z})$ ,  $\bar{c} \neq 0$ , and  $P$  is a principal  $SO(3)$ -bundle corresponding to a  $U(2)$ -bundle  $\hat{P}$  with  $c_1(\hat{P}) = c$  and  $c_2(\hat{P})$  sufficiently large, then the polynomial  $\Phi_{c, \alpha, P}(X)$  is nontrivial.*

Now suppose that  $X$  has a big monodromy group in the sense of Friedman and Morgan [FMM]. Then the  $SU(2)$ -invariants  $\Phi_l(X)$  of  $X$  are complex polynomials in the canonical class  $k_X$  and the quadratic form  $q_X$  [FMM]. In the  $SO(3)$ -case one finds the following result:

**THEOREM 4.** *Let  $X$  be a simply connected algebraic surface with  $p_g(X) > 0$ ,  $w_2(X) \neq 0$ , and with a big monodromy group. Then, for a principal  $SO(3)$ -bundle  $P$ ,*

$$\Phi_{k_X, \alpha, P}(X) \in \mathbf{C}[k_X, q_X].$$

**COROLLARY 5.** *Let  $X$  be a simply connected algebraic surface with  $p_g(X) > 0$  and with a big monodromy group. Then  $\{\pm k_X\}$  is invariant under  $\psi(\text{Diff}_+(X))$ , if  $k_X$  divides a nontrivial polynomial invariant.*

The corollary follows from the fact that if  $k_X$  divides a nontrivial polynomial invariant, then it is its only linear factor up to multiples (cf. [FMM]).

When are the assumptions of Corollary 5 satisfied? It follows from Theorem 1 that the surfaces listed in this theorem have big monodromy.

Let  $X$  be any simply connected algebraic surface with a big monodromy group. If  $p_g(X) \equiv 0 \pmod{2}$  then the degree of  $\Phi_l(X)$  is odd. If  $p_g(X) \equiv 1 \pmod{2}$  and  $k_X^2 \equiv 1 \pmod{2}$  then the degree of  $\Phi_{k_X, \alpha, P}(X)$  is odd. So  $k_X$  divides  $\Phi_l(X)$  or  $\Phi_{k_X, \alpha, P}(X)$  in these cases.

*Remark.* Theorem 4 and its corollary remain true for polynomials  $\Phi_{c, \alpha, P}(X)$  if  $c \in H^2(X, \mathbf{Z})$  is a class with  $\bar{c} \neq 0$  such that  $\overline{f^*(c)} = \bar{c}$  for all  $f \in \psi(\text{Diff}_+(X))$ . The question which elements of  $H^2(X, \mathbf{Z})$  or  $H^2(X, \mathbf{Z}/2)$  have this invariance property will be treated in §4.

### 3. NON-REALIZABLE ISOMETRIES

We shall show that for a simply connected algebraic surface with odd geometric genus,  $-1$  is not induced by an orientation preserving diffeomorphism. For K3 surfaces this was shown by Donaldson in the proof of [D, Proposition 6.2]. There he proves the nontriviality of a certain polynomial  $\Phi_{c, \alpha, P}(X)$  for a K3 surface  $X$ . With Zuo's nontriviality result (Theorem 3) we are able to generalize this as follows.

**THEOREM 6.** *If  $X$  is a simply connected algebraic surface with  $p_g(X) \equiv 1 \pmod{2}$  then  $-1 \notin \psi(\text{Diff}_+(X))$ .*

*Proof.* Suppose that there is an orientation preserving diffeomorphism  $f: X \rightarrow X$  such that  $f^* = -1$ . Let  $c \in H^{1,1}(X, \mathbf{Z})$  be a class with  $\bar{c} \neq 0$ , and choose a principal  $SO(3)$ -bundle  $P$  with  $w_2(P) = \bar{c}$  such that  $\Phi_{c, \alpha, P}(X)$  is nontrivial. This is possible according to Theorem 3. Then

$$f^* \Phi_{c, \alpha, P}(X) = (-1)^d \Phi_{c, \alpha, P}(X),$$

since  $\Phi_{c, \alpha, P}(X)$  is a polynomial of degree  $d$  on  $L$ .

On the other hand, by §2(c)

$$f^* \Phi_{c, \alpha, P}(X) = \Phi_{f^*c, f^*\alpha, f^*P}(X).$$

We have  $f^*c = -c$  and  $f^*\alpha = -\alpha$  because  $f^* = -1$  and the dimension of  $\alpha$  is odd. Since  $f$  is orientation preserving and  $f^* = -1$  we find  $f^*p_1(P) = p_1(P)$  and  $f^*w_2(P) = w_2(P)$ , so that the bundle  $f^*P$  is isomorphic to  $P$ . Therefore