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Kapitel: 3. NON-REALIZABLE ISOMETRIES

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Let X be any simply connected algebraic surface with a big monodromy group. If $p_g(X) \equiv 0 \pmod 2$ then the degree of $\Phi_l(X)$ is odd. If $p_g(X) \equiv 1 \pmod 2$ and $k_X^2 \equiv 1 \pmod 2$ then the degree of $\Phi_{k_X, \alpha, P}(X)$ is odd. So k_X divides $\Phi_l(X)$ or $\Phi_{k_X, \alpha, P}(X)$ in these cases.

Remark. Theorem 4 and its corollary remain true for polynomials $\Phi_{c, \alpha, P}(X)$ if $c \in H^2(X, \mathbf{Z})$ is a class with $\bar{c} \neq 0$ such that $\overline{f^*(c)} = \bar{c}$ for all $f \in \psi(\mathrm{Diff}_+(X))$. The question which elements of $H^2(X, \mathbf{Z})$ or $H^2(X, \mathbf{Z}/2)$ have this invariance property will be treated in §4.

3. Non-realizable isometries

We shall show that for a simply connected algebraic surface with odd geometric genus, -1 is not induced by an orientation preserving diffeomorphism. For K3 surfaces this was shown by Donaldson in the proof of [D, Proposition 6.2]. There he proves the nontriviality of a certain polynomial $\Phi_{c, \alpha, P}(X)$ for a K3 surface X. With Zuo's nontriviality result (Theorem 3) we are able to generalize this as follows.

Theorem 6. *If* X *is a simply connected algebraic surface with* $p_g(X) \equiv 1 \pmod 2$ *then* $-1 \notin \psi(\mathrm{Diff}_+(X))$.

Proof. Suppose that there is an orientation preserving diffeomorphism $f: X \to X$ such that $f^* = -1$. Let $c \in H^{1,1}(X, \mathbf{Z})$ be a class with $\bar{c} \neq 0$, and choose a principal $SO(3)$-bundle P with $w_2(P) = \bar{c}$ such that $\Phi_{c, \alpha, P}(X)$ is nontrivial. This is possible according to Theorem 3. Then

$$f^* \Phi_{c, \alpha, P}(X) = (-1)^d \Phi_{c, \alpha, P}(X) ,$$

since $\Phi_{c, \alpha, P}(X)$ is a polynomial of degree d on L.

On the other hand, by §2 (c)

$$f^* \Phi_{c, \alpha, P}(X) = \Phi_{f^*c, f^*\alpha, f^*P}(X) .$$

We have $f^*c = -c$ and $f^*\alpha = -\alpha$ because $f^* = -1$ and the dimension of α is odd. Since f is orientation preserving and $f^* = -1$ we find $f^*p_1(P) = p_1(P)$ and $f^*w_2(P) = w_2(P)$, so that the bundle f^*P is isomorphic to P. Therefore

$$f^*\Phi_{c,\alpha,P}(X) = \Phi_{-c,-\alpha,P}(X) .$$

Applying §2(a) and (b) with $a = -c$ we get

$$f^*\Phi_{c,\alpha,P}(X) = -\Phi_{-c,\alpha,P}(X) = -(-1)^{c^2}\Phi_{c,\alpha,P}(X) .$$

By assumption $\Phi_{c,\alpha,P}(X) \not\equiv 0$, so we must have

$$(-1)^{c^2+1} = (-1)^d .$$

Now $d = 4c_2 - c^2 - 3(1 + p_g(X))$ implies that $(-1)^{p_g(X)} = 1$, i.e. $p_g(X) \equiv 0 \pmod 2$. This proves Theorem 6.

4. Diffeomorphism groups of some algebraic surfaces

In §1 we saw that the image of ψ contains the group $O'_k(L) \cdot \{\sigma_*, \mathrm{id}\}$ in many cases. In §2 we showed that under certain conditions $\{\pm k_X\}$ is invariant under $\psi(\mathrm{Diff}_+(X))$. Finally we proved in the previous section that for algebraic surfaces of odd geometric genus -1 is not induced by an orientation preserving diffeomorphism. It turns out that these facts suffice to determine the image of ψ.

PROPOSITION 7. *Let X be a simply connected algebraic surface which satisfies the following conditions:*

(i) $O'_k(L) \cdot \{\sigma_*, \mathrm{id}\} \subset \psi(\mathrm{Diff}_+(X))$,

(ii) $\{\pm k_X\}$ *is invariant under* $\psi(\mathrm{Diff}_+(X))$,

(iii) $-1 \notin \psi(\mathrm{Diff}_+(X))$.

Then

$$\psi(\mathrm{Diff}_+(X)) = O'_k(L) \cdot \{\sigma_*, \mathrm{id}\} .$$

Proof. Let $g = -\sigma_*$. Then $g \in O_k(L)$, but $g \notin \psi(\mathrm{Diff}_+(X))$, since $-1 \notin \psi(\mathrm{Diff}_+(X))$. Hence by (i), $g \notin O'_k(L)$. Therefore

$$O_k(L) = O'_k(L) \cdot \{g, \mathrm{id}\} .$$

Now let $h \in \psi(\mathrm{Diff}_+(X))$. By (ii) either $h(k) = k$ or $h(k) = -k$. In the first case $h \in O_k(L)$. Moreover, $h \in O'_k(L)$ since otherwise $h = gh_0$ for some $h_0 \in O'_k(L)$ which would imply $g \in \psi(\mathrm{Diff}_+(X))$, a contradiction. In the second case we have $h' = h\sigma_* \in O_k(L)$. By the same argument as before we see that $h' \in O'_k(L)$. Hence $h = h'\sigma_* \in O'_k(L) \cdot \{\sigma_*, \mathrm{id}\}$. This proves Proposition 7.

Putting everything together we get the main result of our paper.