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AUTOMORPHIC SPECTRA ON THE TREE OF PGL_2

by Isaac EFRAT [1])

0. INTRODUCTION

Our aim in this paper is to give a complete development of the spectral theory of functions on the Bruhat-Tits building attached to PGL_2 of a function field over a finite field, which are automorphic with respect to the associated modular group. This set-up may be viewed as the simplest, nontrivial case to which the theory of automorphic forms on GL_2 ([JL]) applies. Using only elementary means, we derive an explicit description of the resulting theory, with emphasis on the underlying spectral decomposition and the distinction between discrete and continuous spectra. This approach has in turn been instrumental in our recent work on the existence problem of cusp forms and their deformation theory ([E1], [E2]).

To describe this set-up in some detail, let k be the finite field with q elements. The norm at infinity of the field of rational functions $k(t)$ is given by

$$| f/g | = q^{\deg(f) - \deg(g)} ,$$

where f and g are polynomials in $k[t]$. The completion with respect to this norm is the field K of Laurent series in t^{-1}

$$\sum_{n=-N}^{\infty} a_n t^{-n} , \qquad a_n \in k ,$$

and those for which $N \geqslant 0$ form the maximal compact subring O of the local integers in K.

Consider the group $G_K = PGL_2(K)$ of all 2×2 invertible matrices over K modulo the scalar matrices. The subgroup

$$G_O = PGL_2(O)$$

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is a maximal compact subgroup, which gives rise to the metric space

$$X = G_K / G_O$$

on which G_K acts via isometries. Thus the subgroup Γ of all elements with polynomial entries is a discrete subgroup that acts on X, resulting in a quotient $F = \Gamma \backslash X$. By an automorphic function we mean a Γ-invariant function on X, which is therefore just a function on F.

There is a natural operator T on finitely supported functions on X, which generates the Hecke algebra of operators that commute with the isometries of X. This operator has been studied by Cartier ([C1], [C2]) with emphasis on its spherical and harmonic functions. Here we bring Γ into play and study the spectral theory of T as an operator on $L^2(F)$. Specifically, it is our aim to give an explicit basis of $L^2(F)$ consisting of automorphic functions that are eigenfunctions of T.

This set-up stands in precise analogy with the classical situation of the modular group $SL_2(\mathbf{Z})$ acting on the upper half plane H. The operator T is then analogous to the Laplace-Beltrami operator of H. It is known (see [I], [T] for expositions) that in this case we have a decomposition into invariant subspaces

$$L^2(F) = R \oplus C \oplus E\,.$$

Here $R \oplus C$ (resp. E) is the subspace spanned by discrete (resp. continuous) eigenfunctions. More precisely, R is simply the one-dimensional space of constant functions. C is the span of the non-constant discrete eigenfunctions, which here are all cusp forms, meaning that they decay rapidly at the cusp of $SL_2(\mathbf{Z}) \backslash H$. This space can be shown to be infinite dimensional. Lastly, E is generated by functions $E(z, 1/2 + it)$ where $E(z, s)$ with $z \in H$, $s \in \mathbf{C}$ is the Eisenstein series attached to $SL_2(\mathbf{Z})$.

Bearing this analogy in mind, our main results are:

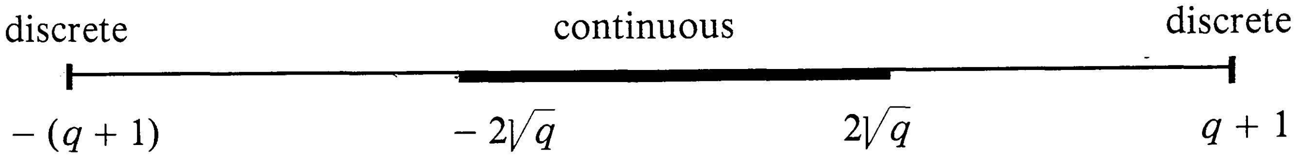
1. *The discrete eigenfunctions of* T *generate a two-dimensional subspace* R, *explicitly given by Proposition 3.5. In particular,* Γ *admits no cusp forms.* This is a special case of much more general dimension formulae (see [D], [HLW], [Sch]).

2. *The eigenfunctions in the continuous spectrum span a subspace* E *which is an isometric image of* $L^2([0,\pi])$ *with respect to the measure*

$$\frac{1}{2\pi}\left((q-1)^2 + 4q\sin^2\theta\right)d\theta$$

(Theorem 4.1).

3. *The above describes a decomposition* $L^2(F) = R \oplus E$ *(Theorem 5.1), made explicit in Theorem 5.3 (compare* [L]*). In particular, the spectrum of* T *on* $L^2(F)$ *is*



1. The tree of $PGL_2(K)$

The material in this section is adapted from Serre [S] and Weil [W].

An O-lattice in K^2 is a set

$$L = \{\alpha v_1 + \beta v_2 \mid \alpha, \beta \in O\}$$

with v_1, v_2 a basis for K^2. We can associate to L the matrix $(v_1, v_2) \in GL_2(K)$ and different choices of bases v_1, v_2 will give cosets in $GL_2(K)/GL_2(O)$. Two lattices L and L' are said to be equivalent if $L' = aL$ for some $a \in K^\times$. We thus have a natural correspondence between equivalence classes of O-lattices in K^2 and points in X.

We define a graph structure on X. Let Λ and Λ' be two equivalence classes of lattices. We say that Λ and Λ' are adjacent if there exist representatives $L \in \Lambda, L' \in \Lambda'$ such that

$$L' \subset L \quad \text{and} \quad L/L' \cong k \,. \tag{1}$$

THEOREM 1.1 ([S]). *The graph whose set of vertices is* X *and whose edges are the pairs* (Λ, Λ') *satisfying (1) is the (infinite)* $(q+1)$*-regular tree.*

We seek a more explicit realization of X. Let B be the Borel subgroup of G_K consisting of the matrices whose bottom row is $(0, 1)$. Then the Iwasawa decomposition is

$$G_K = BG_O \,,$$

but it is not difficult to see that in fact any coset in X has a representative of the form

$$\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$$

with $x \in K$ and a uniquely determined $n \in \mathbf{Z}$.