

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 37 (1991)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: AUTOMORPHIC SPECTRA ON THE TREE OF PGL_2
Autor: Efrat, Isaac
Kapitel: 2. The operator T
DOI: <https://doi.org/10.5169/seals-58728>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 14.03.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

In these coordinates we can write down the $q + 1$ vertices of X that are adjacent to a typical vertex $\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$. They are

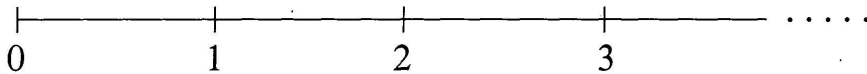
$$\begin{pmatrix} t^{n+1} & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t^{n-1} & \xi t^n + x \\ 0 & 1 \end{pmatrix}, \quad \xi \in k.$$

The group G_K acts on the tree X as a group of automorphisms. We can therefore define a graph structure on the quotient F for the action of Γ on X .

THEOREM 1.2 ([S], [W]). *The quotient graph $F = \Gamma \backslash X$ is given by (the cosets of)*

$$\begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \quad n \geq 0,$$

so that F is the tree



In fact, the vertex $\begin{pmatrix} t^n & x \\ 0 & 1 \end{pmatrix}$ corresponds to n , and so if $n \geq 1$, its neighbor $\begin{pmatrix} t^{n+1} & x \\ 0 & 1 \end{pmatrix}$ corresponds to $n + 1$ while the other q neighbors are represented by $n - 1$. If $n = 0$, all neighbors correspond to 1.

2. THE OPERATOR T

Let μ be the Haar measure on G_K normalized so that $\mu(G_O) = q(q - 1)$. We compute the measure of F induced from μ . Since

$$F = \Gamma \backslash X = \Gamma \backslash G_K / G_O$$

we have

$$\Gamma \backslash G_K = \cup_{s \in F} sG_O,$$

where

$$sG_O = \{\Gamma su \mid u \in G_O\} \subset \Gamma \backslash G_K.$$

The point measure at s will be the measure of sG_O in the quotient space $\Gamma \backslash G_K$. Now we have a correspondence

$$sG_0 \simeq s^{-1}\Gamma_s s \setminus G_0 ,$$

where $\Gamma_s = \Gamma \cap sG_0s^{-1}$ is the finite subgroup of Γ that stabilizes s . Thus

$$\mu(sG_0) = \frac{\mu(G_0)}{|\Gamma_s|} .$$

It is not hard to check that if $s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $|\Gamma_s| = q(q^2 - 1)$, while

for $s = \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}$, $n \geq 1$, $|\Gamma_s| = (q - 1)q^{n+1}$. We therefore put mass

$\frac{1}{q+1}$ at the vertex 0 and q^{-n} at the vertices $n = 1, 2, \dots$, so that if f and g are functions on $F = \{0, 1, 2, \dots\}$ then

$$(2) \quad \langle f, g \rangle = \int_F f \bar{g} d\mu = \frac{1}{q+1} f(0)\bar{g}(0) + \sum_{n=1}^{\infty} f(n)\bar{g}(n)q^{-n} .$$

The algebra of operators on functions on the tree X that commute with the automorphisms of X is generated by the operator

$$(Tf)(s) = \sum_{s' \text{ is adjacent to } s} f(s')$$

(see [C2]). The operator $(q+1)I - T$ is the Laplacian on X .

If f is Γ -automorphic, and therefore can be thought of as a function on F , then T operates on f by

$$(3) \quad (Tf)(n) = \begin{cases} qf(n-1) + f(n+1), & \text{if } n \geq 1, \\ (q+1)f(1), & \text{if } n = 0. \end{cases}$$

PROPOSITION 2.1. T is a self-adjoint operator on $L^2(F)$ with respect to the measure μ .

Proof. If the series $\|f\|^2$ converges, then Cauchy's inequality implies that the four series in $\|Tf\|^2$ also converge. Thus T maps $L^2(F)$ into itself. Now

$$\begin{aligned} \langle Tf, \bar{g} \rangle &= \frac{1}{q+1} (q+1)f(1)g(0) \\ &+ \sum_{n=1}^{\infty} (qf(n-1)g(n) + f(n+1)g(n))q^{-n} \end{aligned}$$

$$\begin{aligned}
&= f(1)g(0) + \sum_{n=0}^{\infty} qf(n)g(n+1)q^{-(n+1)} + \sum_{n=2}^{\infty} f(n)g(n-1)q^{-(n-1)} \\
&= f(1)g(0) + f(0)g(1) + \sum_{n=1}^{\infty} f(n)g(n+1)q^{-n} \\
&\quad + q \sum_{n=1}^{\infty} f(n)g(n-1)q^{-n} - f(1)g(0) \\
&= f(0)g(1) + \sum_{n=0}^{\infty} (f(n)qg(n-1) + f(n)g(n+1))q^{-n} = \langle f, T\bar{g} \rangle .
\end{aligned}$$

3. EIGENFUNCTIONS

An automorphic eigenfunction of T on X with eigenvalue λ is a function on F that satisfies

$$\lambda f(0) = (q+1)f(1) ,$$

$$\lambda f(n) = qf(n-1) + f(n+1) , \quad n \geq 1 .$$

If we write $u(n) = \begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix}$ and normalize $u(0) = \begin{pmatrix} \lambda \\ q+1 \end{pmatrix}$, we obtain the recursion

$$u(n) = A^n u(0)$$

with

$$A = \begin{pmatrix} \lambda & -q \\ 1 & 0 \end{pmatrix} .$$

Let $x_1, x_2 = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 - 4q})$ be the characteristic roots of A and assume that $x_1 \neq x_2$, i.e., that $\lambda \neq \pm 2\sqrt{q}$. Solving the recursion we get

PROPOSITION 3.1. *The eigenfunctions on F with eigenvalue λ are the multiples of the function*

$$(4) \quad f_\lambda(n) = \begin{cases} \frac{1}{x_1 - x_2} (\lambda(x_1^n - x_2^n) - q(q+1)(x_1^{n-1} - x_2^{n-1})) , & \text{if } n \geq 1 \\ q+1 & \text{if } n = 0 . \end{cases}$$