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$$(1.12)\quad \tilde{\zeta}_{x_i x_j \dots x_k} = \zeta_i x_j \dots x_k - \frac{l_j}{l_i} \zeta_j x_i \dots x_k + \cdots \pm \frac{l_k}{l_i} \zeta_k x_i x_j \dots x_{k-1} \in A(Q) .$$

Here is our main result.

THEOREM B. *For a finite abelian group Q, the association*

$$(1.13)\qquad \zeta_{x_i x_j \dots x_k} \mapsto \tilde{\zeta}_{x_i x_j \dots x_k} \in A(Q)$$

identifies $H^*(Q, \mathbf{Z})$ with the graded subalgebra of $A(Q)$ generated (as an algebra) by the $\tilde{\zeta}_{x_i x_j \dots x_k} \in A(Q)$.

This description of the cohomology ring should be compared with that in Chapman [5]. Our description of the cohomology ring has its advantages and disadvantages. An advantage is that it arises from a "small" model; indeed, modulo a prime p dividing each torsion coefficient l_i, our model boils down to the cohomology ring itself. To have a model as small as possible is important for explicit computations. A disadvantage of our description is that it is natural in the *presentation only,* and not in the group itself. Invariant descriptions of the *homology* of a finitely generated abelian group have been given by Hamsher [12] and Decker [6] in their Chicago ph. d. theses supervised by S. Mac Lane. We do not known whether invariant descriptions of the *cohomology* of a finitely generated abelian group have ever been worked out.

I am indebted to S. Mac Lane for discussions and for a number of comments about an earlier version of the paper.

2. THE PROOF OF THEOREM A

To make the paper self contained we reproduce the following material from our paper [13]. The contents of the present Section are of course classical but apparently not as well known as they deserve.

Let R be a commutative ring with 1. By a *Hopf algebra* $(H, \mu, \Delta, \eta, \varepsilon)$ over R we mean as usual a module H together with the structures (μ, η) and (Δ, ε) of an algebra and a coalgebra that are compatible, that is, μ is a morphism of coalgebras or, equivalently, Δ is a morphism of algebras; see e.g. VI.9 in Mac Lane [23]. We mention in passing that some authors call this a bialgebra and require a Hopf algebra to have the additional structure of what is called an *antipode.* For us this is actually of no account since this additional structure will always be present, but there is no need to spell it out. Examples of Hopf algebras are group rings, exterior Hopf algebras, divided polynomial Hopf algebras, polynomial Hopf algebras, see e.g. Mac Lane [23]. Henceforth we write $\Lambda[\upsilon]$ for the exterior Hopf algebra on a single generator υ. As a coalgebra, the divided polynomial Hopf algebra is the symmetric coalgebra over its primitives. For the reader's convenience, and to introduce notation, we recall that, for $i \geqslant 1$, the *divided polynomial Hopf algebra* $\Gamma = \Gamma[u]$ *on* a single generator u of even degree has a generator $\gamma_i(u)$ of degree $|\gamma_i(u)| = i|u|$, subject to the relations

$$\gamma_i(u)\gamma_j(u) = \binom{i+j}{j}\gamma_{i+j}(u)\,, \qquad i, j \geqslant 1\,.$$

Furthermore, the rule

$$\Delta\gamma_i(u) = \sum_{j+k=i} (\gamma_j(u)) \otimes (\gamma_k(u))\,, \qquad i \geqslant 1\,,$$

yields a diagonal map Δ on Γ which endows the latter with the structure of a graded Hopf algebra. The span M of u in $\Gamma[u]$ is the module of *primitives,* and, as a graded coalgebra, Γ is the *symmetric* coalgebra over M. Notice that if R has characteristic zero, the diagonal map Δ is uniquely determined by the requirements that (i) Γ is a Hopf algebra, and that (ii) u primitive.

Let C be an *infinite* cyclic group, pick a generator y, and let $\Lambda[\upsilon_y]$ be the exterior Hopf algebra on a single element υ_y of degree 1. We mention that υ_y may be identified with the suspension of $y - 1 \in RC$, cf. Section 16 of Eilenberg-Mac Lane [9.I], exposé 6 of Cartan [3], but we shall not need this fact. It is well known that the standard small free resolution of R in the category of right RC-modules may be written as a differential graded Hopf algebra

$$\text{(2.1)} \qquad M(C) = (\Lambda[\upsilon_y] \otimes RC, d, \mu, \Delta, \varepsilon, \eta) = (M^{\#}(C) \otimes_d RC, \mu, \Delta, \varepsilon, \eta)\,.$$

Here as a graded commutative algebra, $M^{\#}(C) = \Lambda[\upsilon_y]$, and the underlying graded commutative algebra of $M(C)$ has the tensor product structure, that is, it looks like $M^{\#}(C) \otimes RC$; henceforth we shall discard the tensor product symbol and write $\upsilon_y \otimes 1 = \upsilon_y$ etc. Furthermore, the other structure maps d, Δ, ε, and η are given by the well known formulas

$$\text{(2.2)} \qquad \begin{gathered} d(\upsilon_y) = (y-1)\,, \quad \varepsilon(y^k) = 1 \in R,\, k \geqslant 0\,, \quad \eta(1) = 1 \in RC\,, \\ \Delta(y) = y \otimes y\,, \quad \Delta(\upsilon_y) = \upsilon_y \otimes y + 1 \otimes \upsilon_y\,. \end{gathered}$$

Notice that the diagonal map Δ is *not* cocommutative.

Likewise, for a *finite* cyclic group $C_l = \langle y; y^l = 1\rangle$, the standard small free resolution of R in the category of right RC_l-modules may be written as an augmented and coaugmented differential graded algebra with diagonal (in the sense of Section 1 of [13])

$$
\begin{aligned}
M(C_l) &= (\Gamma[u_y] \otimes \Lambda[v_y] \otimes RC_l, d, \mu, \Delta, \varepsilon, \eta) \\
&= (M^{\#}(C_l) \otimes_d RC_l, \mu, \Delta, \varepsilon, \eta) .
\end{aligned} \tag{2.3}
$$

Here as a graded commutative algebra, $M^{\#}(C_l) = \Gamma[u_y] \otimes \Lambda[v_y]$ with the tensor product structure, and the underlying graded commutative algebra of $M(C_l)$ has the tensor product structure, too, that is, it looks like $M^{\#}(C_l) \otimes RC_l$; as above we shall henceforth discard the tensor product symbol and write $u_y \otimes 1 = u_y$ etc. Furthermore, the other structure maps d, Δ, ε, and η are given by the well known formulas

$$
\begin{aligned}
d(\gamma_j(u_y)) &= (\gamma_{j-1}(u_y))v_y(1+y+\cdots+y^{l-1}), \\
d((\gamma_j(u_y))v_y) &= (\gamma_j(u_y))(y-1), \\
\Delta(y) &= y \otimes y , \\
\Delta(\gamma_j(u_y)) &= \sum_{0 \leqslant i \leqslant j} \Bigg(\gamma_i(u_y) \otimes \gamma_{j-i}(u_y) \\
&\quad + \sum_{0 \leqslant m < n \leqslant l-1} (\gamma_i(u_y))\, v_y y^m \otimes (\gamma_{j-i-1}(u_y))\, v_y y^n \Bigg) \qquad (2.4) \\
\Delta((\gamma_j(u_y))v_y) &= \sum_{0 \leqslant i \leqslant j} ((\gamma_i(u_y))v_y \otimes (\gamma_{j-i}(u_y))y \;+\; (\gamma_i(u_y)) \otimes (\gamma_{j-i}(u_y))v_y) , \\
\varepsilon(y^k) &= 1 \in R, 0 \leqslant k < l , \qquad \eta(1) = 1 \in RC_l ,
\end{aligned}
$$

where $j \geqslant 0$ or $j \geqslant 1$ as appropriate. Notice that the diagonal map Δ is no longer coassociative. We indicated in [13] (3.2.14) and (3.4.4) how these formulas can be obtained from scratch.

We now write $\bar{M}(C) = M(C) \otimes_{RC} R$ and $\bar{M}(C_l) = M(C_l) \otimes_{RC_l} R$; these are the corresponding *reduced objects* [13]. It is clear that, as differential graded commutative algebras, they look like

(2.5) $\bar{M}(C) = \Lambda[v_y]$ with zero differential,

(2.6) $\bar{M}(C_l) = (\Gamma[u_y] \otimes \Lambda[v_y], d)$, where $\begin{cases} d(\gamma_j(u_y)) = l(\gamma_{j-1}(u_y))v_y , \\ d((\gamma_j(u_y))v_y) = 0 . \end{cases}$

Furthermore, it is manifest that $\bar{M}(C) = \Lambda[v_y] = H_*(C, R)$ as Hopf algebras, and that the above diagonal map induces a diagonal map Δ on $\bar{M}(C_l)$ given by

$$
\Delta(\gamma_j(u_y)) = \sum_{0 \leqslant i \leqslant j} \Bigg(\gamma_i(u_y) \otimes \gamma_{j-i}(u_y)
$$

$$(2.7)\qquad \left. + \frac{l(l-1)}{2}(\gamma_i(u_y))v_y \otimes (\gamma_{j-i-1}(u_y))v_y \right),$$
$$\Delta((\gamma_j(u_y))v_y) = \sum_{0 \leqslant i \leqslant j} ((\gamma_i(u_y))v_y \otimes (\gamma_{j-i}(u_y)) + (\gamma_i(u_y)) \otimes (\gamma_{j-i}(u_y))v_y) .$$

Inspection shows that the latter induces in fact the structure of a (non-cocommutative) differential graded Hopf algebra on $\bar{M}(C_l)$.

If $Q = C_{l_1} \times \cdots \times C_{l_n} \times C^m$ is a finitely generated abelian group, written out as a direct product of cyclic groups as indicated, it is clear that, with the obvious action, the object

$$(2.8)\qquad M(Q) = M(C_{l_1}) \otimes \cdots \otimes M(C_{l_n}) \otimes (M(C))^{\otimes m}$$

is a free resolution of R in the category of RQ-modules. Inspection shows that it coincides with the corresponding free resolution introduced by Tate [29]. Moreover, the tensor product structures turn $M(Q)$ into an augmented and coaugmented differential graded commutative algebra with diagonal. Likewise, the reduced object $\bar{M}(Q) = M(Q) \otimes_{RQ} R$ looks like

$$(2.9)\qquad \bar{M}(Q) = \bar{M}(C_{l_1}) \otimes \cdots \otimes \bar{M}(C_{l_n}) \otimes (\bar{M}(C))^{\otimes m} ,$$

and the tensor product structures turn it into a differential graded commutative Hopf algebra. From this one deduces at once a proof of Theorem A.

Remark 2.10. A few historical comments seem in order: The standard small free resolution of a finite cyclic group is due to Eilenberg and Mac Lane and was first published in § 11 of Eilenberg [8]. The reduced objects $\bar{M}(C)$ and $\bar{M}(C_l)$ without the diagonal map were introduced in Ch. III of Eilenberg-Mac Lane [9.II]; furthermore, in the same reference it is proved that for any finitely generated abelian group Q written out as a direct product as above, as a differential graded algebra, the object $\bar{M}(Q)$ is chain equivalent to the reduced bar construction on the group ring RQ. Moreover, I learnt from Mac Lane that in 1952 he and Eilenberg were aware of the diagonal map spelled out above. The resolution for a finite cyclic group with all the above structure appears on p. 252 of Cartan-Eilenberg [4]; there the resolution is written X_L and the names divided polynomial algebra or Hopf algebra do not occur but the structure is given explicitly. Furthermore, the objects $M(C)$, $M(C_l)$, and $\bar{M}(C_l)$ were recognised as *constructions* by Cartan [2], see also [3] and §§ 2 and 3 of Moore [28]; this fact is heavily exploited in our paper [13].